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Resolution is cut-free

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Abstract In this article, we show that the extension of the resolution proof system to deduction modulo is equivalent to the cut-free fragment of the sequent calculus modulo. The result is obtained through a syntactic translation, without using any cut-elimination procedure. Additionally, we show Skolem theorem and inversion/focusing results. Thanks to the expressiveness of deduction modulo, all these results also apply to the cases of higher-order resolution, Peano's arithmetic and Gentzen's LK.
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1 Introduction

The resolution method [Fit96, NS93, Rob65, RV01] is, with tableaux, one of the best practical ways to perform automatic proof search [RV01, RV02, BDD07]. For this reason resolution has been extended to many formalisms [And71] and in our particular case, to deduction modulo [DHK03], which subsumes the higher-order logic [DW03], Zermelo's set theory [DM07], Peano's arithmetic [DW05a] or Pure Type Systems [Bur08] cases.

Deduction modulo is a formalism that integrates computation and deduction, *via* rewrite rules embedded in, usually, first-order logic deduction rules. This framework is flexible, as the choice of the set of rewrite rules is not constrained, and expressive, for instance all the theories aforementioned have a formulation in a first-order setting. Deduction modulo also allows a unified treatment from a theoretical point of view (no more axioms and axiomatic cuts) and from a practical one : the resolution method of [DHK03] presented here, as well as the tableau methods of [BH06, BH07], apply to any set of rewrite rules. The main novelty of deduction modulo is the ability to rewrite *atomic propositions*:

$$x * y = 0 \longrightarrow x = 0 \vee y = 0$$

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This gives to deduction modulo much of its character, and this is the reason for its expressiveness. The drawback is that the theoretical properties of deduction modulo, such as cut elimination, proof normalization or cut-free completeness are more difficult to prove, or even do not hold, as it could be the case for some particular rewrite systems [DW03, Her05a].

1.1 Proof systems modulo

A natural deduction modulo [DW03], a sequent calculus modulo [DHK03], or even a tableau method modulo [Bon04, BH06, BH07], may be defined in a natural way, in intuitionistic or classical settings. In section 2 below we present the version of classical sequent calculus we will use. Defining a resolution method modulo is unfortunately not as straightforward, in particular because resolution works with sets of clauses, and defining a rewrite relation on clauses is a difficult task. This work has been done in [DHK03].

The resolution method in deduction modulo, Extended Narrowing And Resolution (from now ENAR), involves two rules. The first is an extension of the resolution rule: we add unification constraints on the propositions we resolve. The ability to rewrite atomic propositions into non atomic ones accounts for another rule: extended narrowing.

In this article, we are interested in the equivalence between ENAR and sequent calculus modulo: since there are two systems, one must show that they prove the same formulas. This could be done through soundness and completeness of both proof systems with respect to a given semantic, for instance Stuber [Stu01] presents a completeness theorem of ENAR with a positivity condition on the rewrite system. However, this kind of analysis always requires some condition on the rewrite system, at least for ENAR, as discussed below.

We prefer to explore a syntactical proof of equivalence between both systems. Completeness of ENAR with respect to the cut-free fragment of sequent calculus modulo has been proved in [DHK03], as well as soundness, *under the hypothesis that cut is admissible*. Therefore, if the cut rule is redundant or eliminable, which is a key assumption in [DHK03], ENAR is sound and complete. The main goal of this article is to refine this result and to prove soundness with respect to the *cut-free* fragment of sequent calculus.

1.2 Resolution and cut-free sequent calculus

A link between the cut-free fragment of sequent calculus and resolution is strongly suggested by the fact that both system are necessarily consistent: by a simple case analysis of the applicable rules, one can see that this is impossible to find a derivation of the empty sequent $\vdash$ in the cut-free sequent calculus. Similarly, starting from an empty set of clauses, we cannot derive the contradiction (empty clause).

Therefore, proving soundness and completeness of ENAR with respect to sequent calculus with cuts automatically entails that the sequent calculus modulo is *consistent*. This last assertion is a strong theoretical commitment, almost as strong as the cut-elimination theorem, and requires specific assumptions in deduction modulo, since many rewrite systems, even confluent and terminating ones, are inconsistent or do not have the cut elimination property [DW03, Her05b].

To avoid such theoretical issues, one can either assume the cut rule to be eliminable, as it is done in [DHK03], or show soundness and completeness with respect to cut-free sequent calculus, a point left open in [DHK03], which is the approach chosen here. It will turn out that cut-free sequent calculus and resolution modulo match exactly, even when the corresponding sequent calculus modulo does not enjoy the cut-elimination property, which justifies the title of this article.

Similar works have been done for the inverse method [Mas64, DV01, Min88, Min93], a forward-chaining method that extends to many non-classical sequent calculi, such as the intuitionistic, modal or linear ones. The inverse method has close links with resolution [Tam97, dN95]. The link with cut-free sequent calculus is made easier by the absence of Skolemization and clausal transformation [Min88, Min93] (see section 4 below) and when it is performed, then soundness is not considered [Tam97] or leads to proofs with cuts [dN03].

Sometimes resolution methods are referred to as "proofs by cut". In the light of the results obtained here, this appellation can be claimed to be very confusing and misleading. This is due to the fact that the resolution rule looks as follows:

$$\frac{\neg P, \Gamma' \qquad \Gamma, P}{\Gamma, \Gamma'}$$

or using a separator (let us call it $\twoheadrightarrow$) to separate atoms from negated atoms in a clause (positive and negative literals, see section 4) and making negated atoms appearing on the right hand side, without the now irrelevant $\neg$ sign:

$$\frac{\Gamma \twoheadrightarrow P, \Delta \qquad \Gamma', P \twoheadrightarrow \Delta'}{\Gamma, \Gamma' \twoheadrightarrow \Delta, \Delta'}$$

which is the exact shape of a cut rule. But it does *not* have the same status as a sequent-calculus cut. Indeed a proof search in sequent calculus is backward-chaining, so a cut rule (as any other rule) has to be read bottom-up: we search a derivation of some sequent $\Gamma \vdash \Delta$ and for this, we *introduce* a new formula A, and search derivations of the sequents $\Gamma \vdash A, \Delta$ and $\Gamma, A \vdash \Delta$.

On the contrary a resolution step has to be read top-down, which is the exact opposite since resolution is a forward-chaining method. Resolution never introduces new formulas: on the contrary, it eliminates some. We *first* have the two clauses $\Gamma \twoheadrightarrow A, \Delta$ and $\Gamma', A \twoheadrightarrow \Delta'$ and we *eliminate* A to generate $\Gamma, \Gamma' \twoheadrightarrow \Delta, \Delta'$ that we add to our set of clauses to continue the proof search.

Here is a concrete and simple example of the algorithm given in this article. Let us assume we have no rewrite rules for now, so that the modulo part plays no role and we work in usual resolution and sequent calculus. Starting with the set of clauses $\{A\}, \{\neg A, B\}, \{\neg B\}$ we can build the following resolution derivation:

$$\frac{\dfrac{B, \neg A \qquad A}{B} \qquad \neg B}{\Box}$$

that uses two resolution steps. With the standard notations (section 4 below), this derivation can also be written as:

$$\{A\}_1, \{\neg A, B\}_1, \{\neg B\}_2 \hookrightarrow^1 \{B\}_2 \hookrightarrow^2 \Box$$

Transforming step by step this resolution derivation, starting from the right (resolution step number 2), we first get the following cut-free derivation in sequent calculus:

$$\dfrac{\dfrac{}{A, \neg A \vee B, B \vdash B}\text{Axiom}}{A, \neg A \vee B, \neg B, B \vdash}\neg\text{-l}$$

then, the translation of the first resolution step will eliminate B in this derivation by defining it as the resolvent of A and $\neg A \vee B$. The result of the transformation will be:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A, B}\text{Axiom}}{A, \neg A \vdash B}\neg\text{-l} \qquad \dfrac{}{A, B \vdash B}\text{Axiom}}{A, \neg A \vee B \vdash B}\vee\text{-l}}{A, \neg A \vee B, \neg B \vdash}\neg\text{-l}$$

which is a cut-free derivation. Moreover, each resolution steps is reflected by an axiom rule, that is, in a sense made precise by Linear Logic [Gir87], the dual of a cut rule.

1.3 Outline of this article

In the next section we formally introduce the rewrite rules that we allow in deduction modulo and the associated sequent calculus. The version we use is a modified version of the usual and original sequent calculus modulo [DHK01, DHK03], that is why we then show that it is equivalent, under the assumption that the rewrite system is confluent.

In the following section 3 we prove important syntactical properties of invertibility and focusing in the sequent calculus modulo, following similar properties holding in the usual sequent calculus.

We present in section 4 an intermediate system between sequent calculus modulo and ENAR, EIR (standing for Extended Identical Resolution). It is introduced in [DHK03] and proved there sound and complete with respect to ENAR. Note that both ENAR and EIR reduce to usual resolution when the set of rewrite rules is empty.

Section 5 and 6 present technical material to conveniently handle clauses in the cut-free sequent calculus. Then, section 7 shows how to simulate clausal form transformation into cut-free sequent calculus, which seems to be a new result, and section 8 how to simulate resolution steps. Thanks to the version of sequent calculus used here and the results proved in the sections before, those four sections are quite independent from the rewriting part of deduction modulo and can then be read separately.

2 Sequent calculus modulo

2.1 Language and rewriting relation

We assume that the reader is familiar with the notion of first-order language and formula. $\mathbf{y}$ or $\mathbf{t}$ will be used as a shorthand for some $y_1, ..., y_n$ and $t_1, ..., t_m$ respectively, always clear from the context. The connectors we consider here are $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation), $\Rightarrow$ (implication) and the quantifiers are $\forall$ (universal) and $\exists$ (existential) ; some of them are redundant, as usual in classical logic. Atomic predicates of the language will be denoted $P, Q, ...$ while $A, B, ...$ will designate compound formulas. For more details, see for instance [ST96, TvD88, Tak87].

The rewrite rules allowed in deduction modulo are of two kinds: on terms and, more important, on (atomic) formulas. For instance, a rewrite rule such as $P \longrightarrow B \Rightarrow P$ is allowed. This kind of rewrite rule is the core feature of deduction modulo and is the source of its high potentialities.

Definition 1 A term (resp. propositional) rewrite rule is a pair $l \longrightarrow r$ of terms (resp. formulas such that l is atomic), such that the variables of r appear in l. An equational axiom is a pair $l = r$ of terms.

Definition 2 Let $\mathcal{R}$ be a set of propositional rewrite rules and $\mathcal{E}$ a set of term rewrite rules and equational axioms. A formula A $\mathcal{R}$-rewrites (resp. $\mathcal{E}$-rewrites) in one step to A' if and only if:
$A_{|\omega} = \sigma(l)$ and $A' = A[\sigma(r)]_\omega$ for a rule $l \longrightarrow r \in \mathcal{R}$ (resp. $\mathcal{E}$), an occurrence ω in A and some unifier σ. The application of σ to r, which can be compound, is made capture-avoiding.
We write $A \longrightarrow_{\mathcal{R}} A'$ (resp. $A \longrightarrow_{\mathcal{E}} A'$), and $\longrightarrow^*_{\mathcal{R}}$ (resp. $\longrightarrow^*_{\mathcal{E}}$) for its reflexive, transitive closure. $\equiv_{\mathcal{E}}$ is the congruence generated by the rewrite rules and the equational axioms of $\mathcal{E}$.

A formula A $\mathcal{RE}$-rewrites in one step to A' if and only if it exists a proposition B such that the three following conditions are fulfilled: $A \equiv_{\mathcal{E}} B$, $B \longrightarrow_{\mathcal{R}} B'$ and $B' \equiv_{\mathcal{E}} A'$. This is noted $A \longrightarrow B$, the reflexive transitive closure of this relation is noted $\longrightarrow^*$, and if $A \equiv_{\mathcal{E}} A'$ we also write $A \longrightarrow^* A'$. $\equiv$ is the congruence generated by $\mathcal{RE}$.

We write $\Gamma \equiv \Gamma'$ (resp. $\Gamma \longrightarrow^* \Gamma'$ for two sets of formulas Γ, Γ') if and only if $\Gamma = A_1, ..., A_n$ and $\Gamma' = A'_1, ..., A'_n$ and for any i $A_i \equiv A'_i$ (resp. $A_i \longrightarrow^* A'_i$).

Note that *propositional* rewrite rules are oriented in the $\longrightarrow^*$ relation, while term rewrite rules (in $\mathcal{E}$) are not. This can be changed, by defining $\mathcal{E}$ to contain only the equational axioms, pushing term rewrite rules into $\mathcal{R}$. Here we stick to the original formulation [DHK03].

Definition 3 (Confluence) A rewrite system $\mathcal{RE}$ is said to be *confluent* if and only if for any formulas $A \equiv B$, it exists a formula C such that $A \longrightarrow^* C$ and $B \longrightarrow^* C$.

In the remainder of the article, we will *not* suppose that the considered rewrite system is confluent. This assumption is not needed, although we have to assume this if we want to extend our results to ENAR (theorem 3.1 of [DHK03]) and to usual sequent calculus.

2.2 The sequent calculus

Figure 1 presents the version of sequent calculus modulo we use. It has some special features, but it is equivalent (see below and [Her05a]) to any other formulation, provided the rewrite system is confluent.

We assume familiarity with at least one version of the classical sequent calculus [Gen34, Sza69], so that we only briefly recall definitions. A sequent is a pair of multisets (order does not matter, repetition allowed) of formulas. A proof in the sequent calculus is called a *derivation*; it is a tree formed by a finite number of nodes, such that every node is one of the rules of figure 1. A branch linking two nodes has a unique label, being the corresponding *premise*/*conclusion* of the rule; in particular the leaves must be axioms. The formula being decomposed is called the *principal* formula. The height of a derivation is the depth of the associated tree. For more details see for instance [ST96].

As for non-standard features, note that the rules *axiom, weakening, conversion* involve atomic formulas (and for some, empty contexts). This will be a crucial property when reasoning by induction over derivations, since we need any non atomic formula to be properly broken down, in order, first, to avoid some long case distinction, and second, to be able to ensure properties on the height of the derivations. We do not allow the cut rule, which is a critical feature. Moreover, unlike the usual presentation of mixed deduction and rewrite rules as, for instance:

$$\frac{}{A \vdash B} \text{ Axiom, if } A \equiv B$$

we prefer first, to separate rewrite steps from deduction steps and second, to make an orientation of the rewrite rules so that we only allow upward rewriting, and not equivalence. Two formulas $A \equiv_{\mathcal{E}} B$ fit this pattern, as upward rewriting has to be understood on the propositional rules of $\mathcal{R}$ and modulo $\mathcal{E}$ (definition 2).

The conversion rules are rules we should be careful with: they change the nature of formulas (from atomic to non atomic). So for our purpose it is useful to separate structural decomposition of formula from calculations (rewriting), as it is done also in EIR (section 4).

The benefits we reap from this are twofold: we are closer to what happens in EIR and most of the lemmas and propositions in this article have shapes and proofs quite independent from deduction modulo concerns. We only have to consider two additional rules and sometimes even this is not necessary, furthermore other rules are exactly the rules of usual sequent calculus. Of course, from many other standpoints, the sequent calculus of figure 1 is not well-suited and it should be considered as an intermediate system, in the same way EIR (section 4) is an intermediate system between ENAR and sequent calculus.

For a simple example, consider the language where terms are formed with a constant, 0 and s, a unary function symbol, s, and where predicates are *Even*, *Odd* both binary and $\top$, nullary. Consider the following rewrite system:

$$\begin{aligned} \top &\longrightarrow \neg\bot \\ Even(0) &\longrightarrow \top \\ Even(s(x)) &\longrightarrow Odd(x) \\ Odd(s(x)) &\longrightarrow Even(x) \end{aligned}$$

First, one can build the following derivation:

$$\cfrac{\cfrac{\cfrac{}{\bot \vdash} \bot\text{-r}}{\vdash \neg\bot} \neg\text{-r}}{\vdash \top} \text{conv-r}$$

This is an encoding of the $\top$-r rule as a non-primitive rule ($\top$ is not a logical constant, as in [DHK03]). Then, we can build the following proof that 4 is even:

$$\cfrac{\cfrac{\vdots}{\vdash \top}}{\vdash Even(s(s(s(s(0)))))}$$

which is a four-lines derivation, that one could compare with a derivation of the corresponding sequent in usual sequent calculus:

$$\forall x(Even(s(x)) \Rightarrow Odd(x)), \forall x(Odd(s(x)) \Rightarrow Even(x)), Even(0) \vdash Even(s(s(s(s(0)))))$$

Of course, on such a simple example, there is many optimizations of the sequent-calculus proof search since we have only Horn Clauses [Pfe04].

We are able to derive the more usual rules of figure 2, where the atomicity condition on A is dropped. The dashed line means that there is a means to convert a derivation of one sequent into another one, in the sequent calculus presented in figure 1. In particular, one

identity group

$$P \text{ atomic } \frac{}{P \vdash P} \qquad \text{no cut rule}$$

logical group

$$\wedge\text{-l } \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \text{ } \wedge\text{-r}$$

$$\vee\text{-l } \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma A \vee B \vdash \Delta} \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} \text{ } \vee\text{-r}$$

$$\Rightarrow\text{-l } \frac{\Gamma, B \vdash \Delta \qquad \Gamma \vdash A, \Delta}{\Gamma A \Rightarrow B \vdash \Delta} \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta} \text{ } \Rightarrow\text{-r}$$

$$\neg\text{-l } \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \text{ } \neg\text{-r}$$

$$\bot\text{-l } \frac{}{\bot \vdash}$$

$$\forall\text{-l } \frac{\Gamma, \{t/x\}A \vdash \Delta}{\Gamma, \forall x A \vdash \Delta} \qquad \frac{\Gamma \vdash \{c/x\}A, \Delta}{\Gamma \vdash \forall x A, \Delta} \text{ } \forall\text{-r, } c \text{ fresh constant}$$

$$\exists\text{-l, } c \text{ fresh constant } \frac{\Gamma, \{c/x\}A \vdash \Delta}{\Gamma, \exists x A \vdash \Delta} \qquad \frac{\Gamma \vdash \{t/x\}A, \Delta}{\Gamma \vdash \exists x A, \Delta} \text{ } \exists\text{-r}$$

structural group

$$\text{contr-l } \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \text{ contr-r}$$

$$\text{weak-l, } P \text{ atomic } \frac{\Gamma \vdash \Delta}{\Gamma, P \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash P, \Delta} \text{ weak-r, } P \text{ atomic}$$

rewriting group

$$\text{conv-l, } P \text{ atom, } P \longrightarrow^* B \frac{\Gamma, B \vdash \Delta}{\Gamma, P \vdash \Delta} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash P, \Delta} \text{ conv-r, } P \text{ atom, } P \longrightarrow^* B$$

Fig. 1 Sequent calculus modulo

see that rewriting only atomic formulas is as powerful as rewriting any formulas (under the rather weak hypothesis of confluence). As a convention, we will use a dashed line - - - to denote an admissible rule, and a double line ═══ to denote a repetition of rules. Below are the proofs of admissibility.

Lemma 1 (Admissible rules) *Let Γ, Δ be sets of formulas, A, B be formulas.*

1. *if we have a derivation of $\Gamma \vdash \Delta$ then we can build a derivation of $\Gamma, A \vdash \Delta$ (respectively $\Gamma \vdash A, \Delta$).*
2. *we can prove the sequents $\Gamma, A \vdash A, \Delta$ and $\Gamma, \bot \vdash \Delta$.*
3. *if $A \longrightarrow^* B$ and if we have a derivation of $\Gamma, B \vdash \Delta$ (respectively $\Gamma \vdash B, \Delta$) then we can build a derivation of $\Gamma, A \vdash \Delta$ (respectively $\Gamma \vdash A, \Delta$).*
4. *if $\mathcal{RE}$ is confluent, if $A \equiv B$ and if we have a derivation of $\Gamma, B \vdash \Delta$ (respectively $\Gamma \vdash B, \Delta$) then we can build a derivation of $\Gamma, A \vdash \Delta$ (respectively $\Gamma \vdash A, \Delta$).*

Proof We prove each point separately and in the order they are stated. The first point is proved by induction on the structure of A. If A is atomic then the derivation is obtained by a direct application of the weakening rule of figure 1. Otherwise, if $A = B \wedge C$, then by induction hypothesis, we have a derivation of $\Gamma, B \vdash \Delta$, and by induction hypothesis again, a

$$\text{weak-l}\ \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Gamma} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Gamma}\ \text{weak-r}$$

$$\text{axiom}\ \frac{}{\Gamma, A \vdash A, \Delta} \qquad \frac{}{\Gamma, \bot \vdash \Delta}\ \bot\text{-l}$$

$$\text{conv-l}, A \longrightarrow^* B\ \frac{\Gamma, B \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A, \Delta}\ \text{conv-r}, A \longrightarrow^* B$$

$$\text{conv-l}, A \equiv B\ \frac{\Gamma, B \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A, \Delta}\ \text{conv-r}, A \equiv B$$

Fig. 2 Admissible rules of sequent calculus modulo

derivation of $\Gamma, B, C \vdash \Delta$. We apply an $\wedge$-l rule and obtain the desired derivation. To obtain a derivation of $\Gamma \vdash A, \Delta$ we apply the $\wedge$-r rule to the derivations of $\Gamma \vdash B, \Delta$ and $\Gamma \vdash C, \Delta$ obtained by induction hypothesis. All the other cases are similar.

The second point is proved in two steps. First, we show that $A \vdash A$ has a derivation by induction on the structure of A. If A is atomic then the derivation is a direct application of the axiom rule of figure 1. Otherwise, if $A = B \wedge C$, we build the following derivation:

$$\cfrac{\cfrac{\text{weak-l}\ \cfrac{B \vdash B}{B, C \vdash B} \qquad \cfrac{C \vdash C}{B, C \vdash C}\ \text{weak-l}}{B, C \vdash B \wedge C}\ \wedge\text{-r}}{B \wedge C \vdash B \wedge C}\ \wedge\text{-l}$$

using the first point and induction hypothesis to get derivations of $B \vdash B$ and $C \vdash C$. Once $A \vdash A$ is proved, we add the contexts Γ, Δ by repeated weakening, again applying the first point by induction on the cardinality of Γ, Δ.

The third point is proved by generalizing slightly: we show that if $\Gamma \longrightarrow^* \Gamma'$, $\Delta \longrightarrow^* \Delta'$ and $\Gamma' \vdash \Delta'$ has a derivation, then $\Gamma \vdash \Delta$ has a derivation. It is shown by induction on the derivation of $\Gamma' \vdash \Delta'$. If the last rule is a structural rule on A', then we apply induction hypothesis and contract (or weaken) on A. If the last rule is an axiom, then we have to show $A_1 \vdash A_2$ with $A_1 \longrightarrow^* A'$ and $A_2 \longrightarrow^* A'$. Since A_1 and A_2 are atomic (A' is atomic) we can use conv-l and conv-r on them to turn them both into A'. If the last rule is a conversion rule on $A' \longrightarrow^* A''$, then $A \longrightarrow^* A''$ and we apply induction hypothesis.

Otherwise, if the last rule is $\wedge$-l on $A' = A'_1 \wedge A'_2$ and if A is atomic, then rewrite it to $A'_1 \wedge A'_2$. If A is not atomic, skip this step, since A has to be of the shape $A_1 \wedge A_2$ with $A_1 \longrightarrow^* A'_1$ and $A_2 \longrightarrow^* A'_2$, due to the definition of the rewrite relation $\longrightarrow^*$. By induction hypothesis, we have a derivation of $\Gamma, A_1, A_2 \vdash \Delta$, to which we apply the $\wedge$-l rule. All the other remaining cases are dealt with identically.

The last point is also a consequence of a slightly generalized result: we build a derivation of the sequent $\Gamma \vdash \Delta$, assuming that we have a derivation of $\Gamma' \vdash \Delta'$, with $\Gamma \equiv \Gamma'$ and $\Delta \equiv \Delta'$. This proof is done by induction over the structure of the derivation of $\Gamma' \vdash \Delta'$, and heavily uses the confluence of the rewrite system. If the last rule is a contraction or weakening rule, we apply induction hypothesis and the same rule. If the last rule is a conversion rule, we apply induction hypothesis. If the last rule is an axiom, then we must show $A_1 \vdash A_2$ knowing that $A_1 \equiv A' \equiv A_2$. Let B such that $A_1 \longrightarrow^* B$ and $A_2 \longrightarrow^* B$. It exists by confluence and we have a derivation of $B \vdash B$ from the second point. From the third point, we also can build a derivation of $A_1 \vdash A_2$.

Otherwise, if the last rule is $\wedge$-l on $A' = A'_1 \wedge A'_2$ and A is atomic, we rewrite it to $B_1 \wedge B_2$ obtained by confluence. If A is not atomic, we skip this step, since A has to be of the shape $A_1 \wedge A_2$ with $A_1 \equiv A'_1$ and $A_2 \equiv A'_2$, due to the definition of the rewrite relation $\longrightarrow^*$. By induction hypothesis, we have a derivation of $\Gamma, A_1, A_2 \vdash \Delta$, to which we apply the $\wedge$-l rule. All the other remaining cases are treated identically. ∎

Proposition 1 (Proof system equivalence) *Let Γ, Δ be two sets of formulas and $\mathcal{RE}$ be a confluent rewrite system. Let $\Gamma' \equiv \Gamma$ and $\Delta' \equiv \Delta$ be two sets of formulas. Then:*

- *if there is a cut-free derivation π of the sequent $\Gamma \vdash \Delta$ in the system of [DHK03], then there is a derivation π' of the same sequent in the system of figure 1.*
- *if there is a derivation π of the sequent $\Gamma' \vdash \Delta'$ in the system of figure 1, then there is a cut-free derivation π' of the sequent $\Gamma \vdash \Delta$ in the system of [DHK03].*

Proof See [DHK03], page 41, for a presentation of the sequent calculus used there. For the first statement, we show that every rule of the sequent calculus of [DHK03] is admissible, using lemma 1 (and thus confluence). For instance:

$$\wedge\text{-l}, C \equiv A \wedge B \; \frac{\Gamma, A, B \vdash \Delta}{\Gamma, C \vdash \Delta} \quad \text{is replaced by} \quad \dfrac{\dfrac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,\wedge\text{-l}}{\Gamma, C \vdash \Delta}\,\text{conv-l}, C \equiv A \wedge B$$

Conversely, we cannot directly show that the two conversion rules of figure 1 are admissible in the system of [DHK03], so we build π' by induction on the structure of π, which is straightforward and does not involve confluence. Let us work out some key cases:

- if the rule is a conversion rule, then we apply induction hypothesis.
- if the rule is a $\wedge$-l, then we have a derivation of the sequent $\Gamma', A', B' \vdash \Delta'$ and by induction hypothesis, a derivation of the sequent $\Gamma, A', B' \vdash \Delta$ to which we can apply a $\wedge$-l rule to form the corresponding $C \equiv A' \wedge B'$.
- all the other cases are dealt with in the exact same way. ∎

2.3 Some syntactic precision

Definition 4 (Replacement in a derivation) Let Γ, Δ be sets of formulas, and π be a derivation of $\Gamma \vdash \Delta$. Let c be a constant and t be a ground term. We define the *replacement* of c by t in π by induction over π as:

- if no formula of Γ, Δ contains c, then leave π unchanged.
- otherwise, replace c by t in all the derivations of the premises (the immediate subproofs of π), and substitute c by t in all the formulas of Γ, Δ.

So, if the sequent at the root of a subproof π' of π does not mention c, the substitution is not carried out further by a *replacement*: the syntax allows to introduce again c as a fresh constant with an $\exists$-l or a $\forall$-r rule, and those instances of c should not be replaced.

Lemma 2 *Let Γ, Δ be sets of formulas. Let π be a derivation of the sequent $\Gamma \vdash \Delta$, and let c be a constant, t be a ground term. Let π' the replacement of c by t in π.*

If t does no contain constants introduced by $\exists$-l or $\forall$-r rules, π' is a derivation of $\{t/c\}\Gamma \vdash \{t/c\}\Delta$. Otherwise the statement also holds, up to a preliminary replacement of these constants by new fresh ones in the corresponding subproofs of π.

Proof By induction over the structure of π. The proof of the second claim uses the first to show that renaming fresh constants in the subproofs of π is legal. ∎

3 Inversion and focusing in sequent calculus modulo

In this section, we show some important syntactical results on the order in which we can apply rules in sequent calculus modulo. Those results are extensions to deduction modulo and to our particular needs of similar results for sequent calculus [Kle52b]. They answer to the following question: given a provable sequent $\Gamma \vdash \Delta$, can a derivation begin with a logical rule with principal formula some chosen A ? Logical rules can be classified in two groups, *synchronous* (or positive) and *asynchronous* (or negative), such that:

- asynchronous rules can be applied at any moment,
- synchronous rules can be *delayed* and *grouped* until we come across in π an *asynchronous* logical rule on the formula we are breaking down.

This is a *focusing* result, as it can exist in Linear [And92] or in Intuitionistic Logic [LM07, Pfe04, How98]. Similar specialized calculus with a stoup formula also exist for Classical Logic [Her95]. The only minor improvements we need are statements about the height and/or the number of contractions present in the resulting derivation π'. The asynchronous, or invertible, rules are $\vee$-l, $\wedge$-l, $\Rightarrow$-l, $\neg$-l, $\vee$-r, $\wedge$-r, $\Rightarrow$-r, $\neg$-r, $\exists$-l and $\forall$-r whereas the synchronous rules are $\forall$-l and $\exists$-r. By extension, a compound formula A is synchronous (resp. asynchronous) in a sequent $\Gamma \vdash \Delta$ when A is a universal quantification and $A \in \Gamma$, or an existential quantification and $A \in \Delta$ (resp. all the other cases).

Conversion rules have no definitive status in this regard since they apply only to atoms. What one could say is that, under the hypothesis of confluence of the rewrite system, if $A \equiv B$ are two compound formulas then they share the same structure and therefore the same synchronicity. This is not useful here, thanks to the definition of sequent calculus modulo of figure 1, but this could refine a little bit the results presented in this section.

To be more precise, every non-quantifier logical rule is *both* synchronous and asynchronous in the classical case. Only Linear Logic [And92, Gir87] has a sharp distinction between synchronous and asynchronous rules. Here, we are only interested in the asynchronous nature of those rules.

At first sight, conversion rules also have this double characteristic. Things get more complicated in the absence of confluence: application of a rewrite rule may be a real choice point.

Building an efficient proof search system in deduction modulo and reducing nondeterministic choices as much as possible is beyond the scope of this article. So we do not optimize the sequent calculus to behave differently on asynchronous (applied eagerly, without contraction) and on synchronous rules (applied to a single formula until it becomes asynchronous, and at choice points to which we may want to backtrack), as it is done for instance in [Pfe04]. Also, we do not try to embed contraction into logical rules to get a contraction-free system or to get only asynchronous rules [BK07].

All those changes over, and improvements on, sequent calculus are extremely interesting to study and possible in deduction modulo as well. Here it would not be beneficial: links between resolution and sequent calculus would interfere with constraints on the proof system. Even in the case of a contraction-free system, proofs would not be simplified. That is why we stick to the presentation of figure 1, that we claim optimal for our needs, and leave outside the scope of this article an extensive study of such optimized systems, to which the results shown in the next two subsections could however lead.

3.1 Inversion of rules

Invertibility of logical rules has been studied by Kleene in the classical sequent calculus [Kle52b], we extend it here to deduction modulo. A rule r is said to be invertible if each premise is derivable from a proof of the conclusion. That is, whenever a sequent is provable, the premises obtained by applying r to this sequents are provable as well: applying r does not change the property “provable/unprovable” of a sequent. Otherwise stated, if we have, in a derivation of $\Gamma \vdash \Delta$, a rule r on a formula A, we can permute r to the bottom, until we come across the root of the derivation or a logical rule on A, if A appears only as a subformula of some formula of Γ, Δ.

In this article, we look at this lemma from the first point of view: given a derivation of a sequent $\Gamma \vdash \Delta$ containing a compound formula A, we can build a derivation of the premises given by the application to $\Gamma \vdash \Delta$ of a logical rule with A as principal formula.

Lemma 3 (Kleene) *Let Γ, Δ be sets of formulas and A, B be formulas. If we have a derivation π of height h of a sequent:*

- $\Gamma, A \vee B \vdash \Delta$,
- $\Gamma, A \wedge B \vdash \Delta$,
- $\Gamma, \neg A \vdash \Delta$,
- $\Gamma, A \Rightarrow B \vdash \Delta$,
- $\Gamma, \exists x A \vdash \Delta$,
- $\Gamma \vdash A \vee B, \Delta$,
- $\Gamma \vdash A \wedge B, \Delta$,
- $\Gamma \vdash \neg A, \Delta$,
- $\Gamma \vdash A \Rightarrow B, \Delta$,
- $\Gamma \vdash \forall x A, \Delta$,

then we can respectively construct derivations of:

- $\Gamma, A \vdash \Delta$ *and* $\Gamma, B \vdash \Delta$,
- $\Gamma, A, B \vdash \Delta$,
- $\Gamma \vdash A, \Delta$,
- $\Gamma, B \vdash \Delta$ *and* $\Gamma \vdash A, \Delta$,
- $\Gamma, \{c/x\}A \vdash \Delta$ *for any fresh constant* c,
- $\Gamma \vdash A, B, \Delta$,
- $\Gamma \vdash A, \Delta$ *and* $\Gamma \vdash B, \Delta$,
- $\Gamma, A \vdash \Delta$,
- $\Gamma, A \vdash B, \Delta$,
- $\Gamma \vdash \{c/x\}A, \Delta$ *for any fresh constant* c,

Moreover the obtained derivations have a height at most $h - 1$.

Proof All proofs are similar and by induction over h. We work out the $\vee$-left case and point out the places where the proof for other cases differs significantly. This happens only for the quantifier cases.

h cannot be equal to 0 since the last rule cannot be an axiom ($A \vee B$ is not atomic). We now distinguish cases, according to the last rule of π. Suppose this is a rule r applied to a formula of Γ or Δ. We apply induction hypothesis to the premises, and apply rule r to the derivations obtained. For instance, if the rule is $\wedge$-left:

$$\dfrac{\dfrac{\pi'}{\Gamma, C, D, A \vee B \vdash \Delta}}{\Gamma, C \wedge D, A \vee B \vdash \Delta} \wedge\text{-l}$$

Then, by induction hypothesis we get derivations of the sequents:

$$\frac{\pi'_1}{\Gamma, C, D, A \vdash \Delta} \qquad \frac{\pi'_2}{\Gamma, C, D, B \vdash \Delta}$$

We use $\wedge$-left to obtain derivations of the sequents:

$$\Gamma, C \wedge D, A \vdash \Delta \qquad \Gamma, C \wedge D, B \vdash \Delta$$

We check that the statement on height holds.

In the $\exists$-left case, we constrain the induction hypothesis to be applied with the same fresh constant c in the case where two premises appear. Also, a small easily-resolved subtlety arises if r is a quantifier rule. Let c be a fresh constant for the sequent $\Gamma, \forall yC, \exists xA \vdash \Delta$: the term t introduced by a $\forall$-l rule may contain c. So, in the derivation of the premise $\Gamma, \{t/x\}C, \exists xA \vdash \Delta$, we fist replace (in the sense of definition 4) c by d. We get a derivation of $\Gamma, \{t'/x\}C, \exists xA \vdash \Delta$, where $t' = \{d/c\}t$, to which we apply induction hypothesis and then a $\forall$-l rule. The treatment is similar for the three other rules.

Now, for the base cases. If the last rule of π is an $\vee$-left rule with principal formula $A \vee B$:

$$\frac{\dfrac{\pi_1}{\Gamma, A \vdash \Delta} \qquad \dfrac{\pi_2}{\Gamma, B \vdash \Delta}}{\Gamma, A \vee B \vdash \Delta}\ \vee\text{-left}$$

The premises are derivations of what we want (up to replacement of a fresh constant by another in the $\exists$-l rule on $\exists xA$ case), and fit the conditions.

The last case is a contraction rule:

$$\frac{\dfrac{\pi'}{\Gamma, A \vee B, A \vee B \vdash \Delta}}{\Gamma, A \vee B \vdash \Delta}\ \text{contr-l}$$

Then, we apply the induction hypothesis once, and get two derivations of the sequents $\Gamma, A, A \vee B \vdash \Delta$ and $\Gamma, B, A \vee B \vdash \Delta$ of height at most $h-2$. This allows us to apply once again the induction hypothesis, to get derivations of the sequents $\Gamma, A, A \vdash \Delta$ and $\Gamma, B, B \vdash \Delta$ of height at most $h-3$. We get four derivations, but keep only the two that we need. Applying a contraction rule to both of those derivations, we get derivations of the sequents $\Gamma, A \vdash \Delta$ and $\Gamma, B \vdash \Delta$ of height at most $h-2$.

In the $\exists xA$ case, we want to contract a sequent of the shape $\Gamma, \{c/x\}A, \{d/x\}A \vdash \Delta$, for some fresh constant d. Before that, we have to replace d by c. ■

Remark 1 Lemma 2 allows to strengthen easily lemma 3: from a derivation of the sequent $\Gamma, \exists xA \vdash \Delta$ one can construct a derivation of $\Gamma, \{t/x\}A \vdash \Delta$ for any term t.

3.2 Focusing synchronous rules

Lemma 3 above defines the asynchronous rules and formulas. In sequent calculus, the only pure synchronous rules are $\forall$-left and $\exists$-right, that are not invertible. For instance, there is an easy two-step derivation of the sequent $\forall xP \vdash \forall xP$ but we cannot find a term t such that the sequent $\{t/x\}P \vdash \forall xP$ has a derivation, because of the freshness condition on the $\forall$-r rule.

In some versions of the sequent calculus, integrating the contraction rules to the $\forall$-l and $\exists$-r rules [BK07], allows those rules to be invertible as well, and the derivation to begin with a $\forall$-l rule. However, it does not solve the instantiation problem, since translated into the version of sequent calculus of figure 1 the derivation above would be turned into:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\{c/x\}P \vdash \{c/x\}P}}{\{c/x\}P, \{t/x\}P \vdash \{c/x\}P}\,\text{weak-l}}{\forall xP, \{t/x\}P \vdash \{c/x\}P}\,\forall\text{-l}}{\forall xP, \{t/x\}P \vdash \forall xP}\,\forall\text{-r}}{\forall xP, \forall xP \vdash \forall xP}\,\forall\text{-l}}{\forall xP \vdash \forall xP}\,\text{contr-l}$$

t being a dummy term that does not contain c. In fact, this kind of systems does not *permute* the $\forall$-left rule downwards. It is impossible and it is unfortunately what we need here since we will soon need to instantiate universally quantified formulas on the left to a *specific* term. The technique used here is to permute them upward, as far as we can.

Definition 5 (Focused $\forall$-l rules) Let π be a derivation in sequent calculus. We say that π has its *$\forall$-l rules focused* (in short, "π is focused") if and only if for every $\forall$-l rule in π on a formula $\forall xA$, the rule applied on the premiss is a rule on (the newly introduced instance of) A that is not the contraction.

Below are two examples: the left derivation is focused, while the right one is not.

$$\forall\text{-l}\,\cfrac{\exists\text{-l}\,\cfrac{\lor\text{-r}\,\cfrac{\forall\text{-l}\,\cfrac{\forall\text{-l}\,\cfrac{\lor\text{-l}\,\cfrac{\text{ax}\,\cfrac{}{P \vdash P, Q} \qquad \text{ax}\,\cfrac{}{Q \vdash P, Q}}{P \lor Q \vdash P, Q}}{\forall u(P \lor Q) \vdash P, Q}}{\forall z \forall u(P \lor Q) \vdash P, Q}}{\forall z \forall u(P \lor Q) \vdash P \lor Q}}{\exists y \forall z \forall u(P \lor Q) \vdash P \lor Q}}{\forall x \exists y \forall z \forall u(P \lor Q) \vdash P \lor Q}
\qquad
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{P(x,y) \vdash P(x,y)}\,\text{ax}}{\forall y P(x,y) \vdash P(x,y)}\,\forall\text{-l}}{\forall y P(x,y) \vdash \forall y P(x,y)}\,\forall\text{-r}}{\forall x \forall y P(x,y) \vdash \forall y P(x,y)}\,\forall\text{-l}}{\forall x \forall y P(x,y) \vdash \forall x \forall y P(x,y)}\,\forall\text{-r}$$

Note that we can transform the right derivation into a focused derivation of the same sequent. This is the topic of the remainder of the section.

Lemma 4 (Generalization) *Let π be a derivation of the sequent $\Gamma, \{t/x\}A \vdash \Delta$. Assume that π is focused. Then, we can construct a focused derivation π' of the sequent $\Gamma, \forall xA \vdash \Delta$ such that π' possesses the same number of contraction rules as π.*

Moreover, letting r be the last rule of π, if r is a rule on A (not contraction) then the last rules of π' are $\forall$-l on $\forall xA$ and r, otherwise the last rule of π' is also r.

The derivation π' will have almost the same shape as π, but we will make the additional rule $\forall$-l percolate upward. The height might grow, since we add a rule.

Proof By induction over the ordered pair $\langle C(\pi), h(\pi)\rangle$ where h is the height of π and C the number of contractions contained in π. This is the reason for adding a contraction hypothesis in the statement of the lemma. As this type of induction will appear repeatedly in the following proofs, we shall detail it here. An alternative is to introduce an arbitrary number of copies $\forall xA, \ldots, \forall xA$ in the statement of the lemma [Her05a]. In this case, the proof works by induction over the height only.

If the last rule r of π is a rule on Γ or Δ, then we apply induction hypothesis to the derivation(s) of the premise(s) (of lower height), and recombine with r the focused derivation(s) obtained to get a derivation of $\Gamma, \forall xA \vdash \Delta$. Freshness of constants is preserved, in the $\forall$-r and $\exists$-l cases.

Additionally, if r is $\forall$-l on some formula $B \in \Gamma$, then the last rule of the derivation of the sequent $\Gamma, B, \{t/x\}A \vdash \Delta$ is a rule r' on B (not contraction) since π is focused. After the application of induction hypothesis, the last rule of the derivation of $\Gamma, B, \forall xA \vdash \Delta$ is the same rule r' on B. Hence we can safely add a $\forall$-l rule on B, and this derivation π' is focused. This shows the necessity of each hypothesis in the statement of the lemma. If r is not a $\forall$-l rule, then showing that π' is focused is straightforward.

Otherwise, the last rule r of π is a rule on A:

- if it is a contraction, we have a derivation π_1 of the sequent $\Gamma, \{t/x\}A, \{t/x\}A \vdash \Delta$ with $C(\pi_1) = C(\pi) - 1$ contractions. We apply induction hypothesis, and get a derivation of $\Gamma, \forall xA, \{t/x\}A \vdash \Delta$ with $C(\pi_1) = C(\pi) - 1$ contractions. We apply induction hypothesis once more, get a derivation of $\Gamma, \forall xA, \forall xA \vdash \Delta$ with $C(\pi_1) = C(\pi) - 1$ contractions, and we contract on $\forall xA$.
- for any other rule, we add a $\forall$-l rule. We just check that the $\forall$-l rules remain focused. For instance, if it is a $\forall$-l rule on A, π' is:

$$\dfrac{\dfrac{\pi}{\Gamma, \{t/x\}\forall yA \vdash \Delta}\ \forall\text{-l}}{\Gamma, \forall x \forall yA \vdash \Delta}\ \forall\text{-l}$$

  and π' is focused, since π is. If the rule is not $\forall$-l, the argument is the same.

The number of contractions of the obtained derivation is left as it was, because we do not introduce new ones. ∎

Lemma 5 (Regrouping) *Let Γ, Δ be sets of formulas. Let π be a derivation of the sequent $\Gamma \vdash \Delta$. Then we can build a* focused *derivation π' of the same sequent, containing the same number of contractions as π.*

Proof By induction over the height of π, considering the last rule r of π:

- if r is not $\forall$-l, then we apply induction hypothesis on the derivation(s) of the premise(s), and then r to the derivation(s) obtained.
- if r is $\forall$-l on a formula $\forall xA \in \Gamma$, then we have a smaller derivation π_1 of $\Gamma, \{t/x\}A \vdash \Delta$. We apply induction hypothesis to π_1, obtain a derivation π_1' of the same sequent with the $\forall$-l rules focused, and apply lemma 4. ∎

There exists similar results for the other asynchronous rules, constraining them to be focused all together. Lemma 12 can for instance be used (with an empty set of variable J) as an equivalent of lemma 4 and thus as a basis to focus the $\vee$-l rule.

4 Resolution modulo

The resolution method is a refutation methods starting with a set of *clauses*. A clause is a set of (labelled) *literals*, i.e. of atoms and negated atoms, potentially containing free variables. The empty clause (contradiction) is denoted $\square$.

Every formula should be *labelled* with a set of variable names, called the *free variables*, a superset of its actual free variables. We use the labels to perform Skolemization, and labels are generated by the clausal form transformation, see figure 3. This additional step is required because we are in deduction modulo and rewrite rules interfere with variables [DHK03]. Labels have to be considered during substitution and $\mathcal{E}$-equivalence:

Definition 6 Let θ be a substitution and P^l a labelled formula. $(P^l)\theta$ is $(P\theta)^{l'}$, where l' is the set of the free variables of $l\theta$.
Let $Q^{l'}$ be another formula. $P^l \equiv_{\mathcal{E}} Q^{l'}$ if and only if $P \equiv_{\mathcal{E}} Q$ and $l = l'$.

Given a statement $\Gamma \vdash \Delta$ (expressed as in the sequent calculus) to prove, the first task is to transform $\Gamma, \neg\Delta$ into a set of clauses, $cl(\Gamma, \neg\Delta)$, its *clausal (normal) form*. Here and later, we use the following conventions: $\neg\Delta$ represents the set of formulas $\{\neg A \mid A \in \Delta\}$, ψ represents a set of labelled formulas (understood as a disjunction), $\psi \cup \{P^l\}$ is abbreviated to ψ, P^l and Φ is a set of set of formulas (understood as a conjunction). $\Phi \cup \{\psi\}$ is abbreviated as $\Phi \mid \psi$. If ψ contains only labelled literals it is then a clause. If Ψ contains only clauses, it is in clausal form.

The transformation starts with $\Phi = \{A_1\} \mid \cdots \mid \{A_n\} \mid \{B_1\} \mid \cdots \mid \{B_m\}$ where $\Gamma = A_1, \cdots, A_n$ and $\neg\Delta = \{B_1, \cdots, B_m\}$, since $\Gamma, \neg\Delta$ has to be understood as a *conjunction* of formulas. The transformation defined by the rule of figure 3 transforms step by step a set of set of formulas Φ into a set of set of formulas Φ', noted $\Phi \rightsquigarrow \Phi'$. This process is terminating and confluent (proviso a variable and Skolem function renaming), it is standard at the exception of the labels and performs at the same time skolemization, clausification and translation to negation normal form.

$\Phi \mid \psi, (P \wedge Q)^l$	$\rightsquigarrow$	$\Phi \mid \psi, P^l \mid \psi, Q^l$	$\Phi \mid \psi, (\neg(P \vee Q))^l$	$\rightsquigarrow$	$\Phi \mid \psi, (\neg P)^l \mid \psi, (\neg Q)^l$
$\Phi \mid \psi, (P \vee Q)^l$	$\rightsquigarrow$	$\Phi \mid \psi, P^l, Q^l$	$\Phi \mid \psi, (\neg(P \wedge Q))^l$	$\rightsquigarrow$	$\Phi \mid \psi, (\neg P)^l, (\neg Q)^l$
$\Phi \mid \psi, (P \Rightarrow Q)^l$	$\rightsquigarrow$	$\Phi \mid \psi, (\neg P)^l, Q^l$	$\Phi \mid \psi, (\neg(P \Rightarrow Q))^l$	$\rightsquigarrow$	$\Phi \mid \psi, P^l \mid \psi, (\neg Q)^l$
$\Phi \mid \psi, \bot^l$	$\rightsquigarrow$	$\Phi \mid \psi$	$\Phi \mid \psi, (\neg\bot)^l$	$\rightsquigarrow$	Φ
$\Phi \mid \psi, (\forall x P)^{\mathbf{y}}$	$\rightsquigarrow$	$\Phi \mid \psi, P^{\mathbf{y},x}$ (**)	$\Phi \mid \psi, (\neg\exists x P)^{\mathbf{y}}$	$\rightsquigarrow$	$\Phi \mid \psi, (\neg P)^{\mathbf{y},x}$ (**)
$\Phi \mid \psi, (\exists x P)^{\mathbf{y}}$	$\rightsquigarrow$	$\Phi \mid \psi, (\{f(\mathbf{y})/x\}P)^{\mathbf{y}}$ (*)	$\Phi \mid \psi, (\neg\forall x P)^{\mathbf{y}}$	$\rightsquigarrow$	$\Phi \mid \psi, (\neg\{f(\mathbf{y})/x\}P)^{\mathbf{y}}$ (*)
$\Phi \mid \psi, (\neg\neg P)^l$	$\rightsquigarrow$	$\Phi \mid \psi, P^l$			

* : f is a function symbol not appearing in Φ, nor in ψ, nor in P (Skolem symbol).
**: x is a fresh variable.

Fig. 3 Clausal form transformation rules

The resolution method is a proof by contradiction method. Derivation of new clauses is done along the ENAR rules [DHK03] (definition 2.6, page 46). Instead of ENAR, more suited for automated deduction, we use an intermediate system, called EIR (standing for Extended Identical Resolution), introduced in [DHK03]. It is sound and complete with respect to ENAR (proposition 5.1 and 5.2 of [DHK03]). So, any result holding for EIR also holds for ENAR. Moreover, without rewrite rule, EIR, ENAR and resolution [Rob65] collapse.

EIR is technically more convenient to our aim since "logical" inference rules are not yet mixed with "rewrite" inference rules and we do not have unification constraints. All this makes EIR closer to sequent calculus than ENAR.

Inference rules of EIR are presented in figure 4, U refers to a clause, P to a literal and ψ to a set of formulas. Some precisions about labels follow.

$\dfrac{U}{\{x \mapsto t\}U}$	**Instantiation**
$\dfrac{U}{U'}$	**Conversion** if $U \equiv_{\mathcal{E}} U'$
$\dfrac{U, P^{l_1} \quad U', \neg P^{l_2}}{U \cup U'}$	**Identical Resolution**
$\dfrac{U}{U'}$	**Reduction** if $U \longrightarrow_{\mathcal{R}} \psi$ and $U' \in cl(\psi)$

Fig. 4 Inference rules of EIR

1. in the **Instantiation** rule, replace, in all labels, x by the free variables appearing in t.
2. in the **Conversion** rule, labels are the same, according to definition 6.
3. in the **Identical Resolution** rule, labels l_1 and l_2 need not to be the same, although P and $\neg P$ need to be the same.
4. in the **Reduction** rule, the label of ψ are the same than those of U and the label of U' is naturally computed.

Definition 7 (Deduction sequence) Let $\mathcal{RE}$ be a rewrite system. Let $\mathcal{K}$ be a set of clauses and let $U, U_1, \ldots, U_n$ be clauses. $\mathcal{K}, U_1, \ldots, U_n$ is a deduction sequence if and only if for any $p \leq n$, U_p is inferred from clauses in $\mathcal{K}, U_1, ..., U_{p-1}$ using one of the inference rules of figure 4. If there is a sequence starting from $\mathcal{K}$ and ending with U, we abbreviate it as:

$$\mathcal{K} \hookrightarrow_{\mathcal{RE}} U$$

Revisiting the example rewriting system of section 2.2, we can try to show the formula $\forall x(Even(x) \Rightarrow Odd(s(x)))$. The clausal form of the negation of this formula is $Even(c)|\neg Odd(s(c))$ with c a fresh Skolem symbol.

$$\begin{array}{ll} \neg Even(c) & \text{(by \textbf{Reduction})} \\ \Box & \text{(by \textbf{Identical Resolution})} \end{array}$$

We now show the formula $\exists x Odd(x)$. The clausal form of its negation is $\neg Odd(x)$, that we first **instantiate** (for instance to $s(0)$) and then **reduce** directly to the empty clause. As an exercise the reader can also try to show the formula $Even(s(s(s(s(0)))))$, to compare to the derivation of the same statement in section 2.2.

In [DHK03], completeness and soundness of EIR is proved with respect to sequent calculus modulo, under the assumption of cut elimination. The following completeness theorem (proposition 4.2) is then proved:

Theorem 1 (Completeness of EIR [DHK03]) *Let $\mathcal{RE}$ be a rewrite system. Let Γ, Δ be sets of formulas. If we have a cut-free derivation of the sequent $\Gamma \vdash \Delta$, then $cl(\Gamma, \neg\Delta) \hookrightarrow_{\mathcal{RE}} \Box$.*

But the soundness theorem is not the exact converse of this. Given a deduction sequence $cl(\Gamma, \neg\Delta) \hookrightarrow_{\mathcal{RE}} \Box$, it is transformed by proposition 4.1 into a derivation of $\Gamma \vdash \Delta$ with cuts. Of course, all those statements are equivalent when the cut-elimination theorem holds, which is assumed in [DHK03]. We prove a more accurate version, which is the exact converse of theorem 1.

Theorem 2 ((Cut-free) Soundness of EIR) *Let $\mathcal{RE}$ be a rewrite system. Let Γ, Δ be sets of formulas. If $cl(\Gamma, \neg\Delta) \hookrightarrow_{\mathcal{RE}} \Box$ then we can build a* cut-free *derivation of the sequent $\Gamma \vdash \Delta$.*

From this, we will know that, even in the case where cut elimination fails, as might happen in some cases of deduction modulo, resolution and cut-free sequent calculus prove the same statements.

Since EIR is searching for a contradiction (from $\Gamma, \neg\Delta$ we derive $\bot$) and sequent calculus is searching for a direct proof, it should not be surprising that theorem 2 above emulates EIR rules in sequent calculus backward, starting from the – trivial – derivation of $\bot \vdash$ and transforming it into a derivation of $\Gamma, \neg\Delta \vdash$.

5 From clauses to formulas

We already know how to transform formulas into a set of clauses. Here, we perform the reverse operation:

Definition 8 Let $\psi = \{A_1^{l_1}, \ldots, A_n^{l_n}\}$ be a (labelled) set of formulas. Let I an ordered set of n indexes. Let $l = l_1 \cup \ldots \cup l_n$. We define the four following notations:

$$\bigvee_{i \in I} A_i = A_{i_1} \vee (A_{i_2} \vee (\ldots \vee A_{i_n}) \ldots) \qquad \bigvee \psi = \bigvee_{i \in \{1 \ldots n\}} A_i$$

$$\forall^l \mathbf{x} A = \forall x_1 \ldots \forall x_m A \qquad \overline{\psi} = \forall^l \mathbf{x} \bigvee \psi$$

where $\{x_1, \ldots, x_m\} = l$ and l is ordered by a fixed (e.g. alphabetic) order on variable names. If $n = 0$ then we let $\bigvee_{i \in I} A_i = \bot$.

Since a clause is a set of labelled propositions, and since labels are sets, the order in the clauses (and in the labels) is not defined and does not matter. We will see that we can reflect those properties in sequent calculus, as long as we are not interested in the height of the derivations. So in definition 8 above the order of parentheses and the order of quantification will not matter either. Unfortunately, we cannot safely assume this right now: we are forbidden to use the cut rule, and this is the usual way to prove associativity and commutativity. The rest of the section is devoted to that matter, and the order on I and on l is of course relevant for that purpose, until the end of the section.

5.1 $\vee$ is associative-commutative and $\bot$ is neutral

Lemma 6 (Permutations of $\vee$) *Let σ be a n-permutation, and $A_1, \ldots, A_n$ be formulas. If we have a derivation π of the sequent $\Gamma, A_1 \vee (A_2 \vee (\ldots \vee A_n) \ldots) \vdash \Delta$ then we can construct a derivation of the sequent $\Gamma, A_{\sigma(1)} \vee (A_{\sigma(2)} \vee (\ldots \vee A_{\sigma(n)}) \ldots) \vdash \Delta$.*

Proof An informal argument would be: apply Kleene lemma 3 $n-1$ times and recombine the n derivations obtained in the order required by the permutation σ. Formally, the proof is carried out by induction over n, left to the reader. ∎

Remark 2 Applying lemma 6 might cause derivation height to increase up to $n-2$.

As a result of lemma 6, $\vee$ is clearly commutative, since it also turns a derivation of $\Gamma, A \vee B \vdash \Delta$ into a derivation of $\Gamma, B \vee A \vdash \Delta$. In this case, derivation height can even be preserved since we only need to switch some right and left premises.

Lemma 7 (Neutrality of $\bot$) *Let Γ, Δ be two sets of formulas, A be a formula. We have a derivation π of $\Gamma, A \vee \bot \vdash \Delta$ if and only if we have a derivation π' of $\Gamma, A \vdash \Delta$.*

Proof Constructing π' by an easy induction on π for the direct way. For the reverse way, we build the following derivation:

$$\frac{\Gamma, A \vdash \Delta \qquad \overline{\Gamma, \bot \vdash \Delta}}{\Gamma, A \vee \bot \vdash \Delta}$$

∎

5.2 Quantification order does not matter

Lemma 8 (Permuting quantifications) *Let Γ, Δ be sets of formulas and A be a formula. Let σ be a n-permutation. If we have a derivation π of the sequent $\Gamma, \forall x_1 \dots \forall x_n A \vdash \Delta$ then we can build a derivation π' of the sequent $\Gamma, \forall x_{\sigma(1)} \dots \forall x_{\sigma(n)} A \vdash \Delta$ containing the same number of contractions as π.*

Proof From lemma 5, we assume that π is focused. This ensures that all the $\forall$-l rules on $\forall x_1 \dots \forall x_n A$ are applied at the same time, and preserves $C(\pi)$. Then we proceed by induction over the ordered pair $\langle C(\pi), h(\pi) \rangle$, making cases. If the last rule r of π is:

- a rule on a formula of Γ or Δ. We apply the induction hypothesis to the given premise(s) and then r to the derivation(s) obtained.
- a contraction on $\forall x_1 \dots \forall x_n A$. Then, we have a derivation π_1 of the sequent:

  $$\Gamma, \forall x_1 \dots \forall x_n A, \forall x_1 \dots \forall x_n A \vdash \Delta$$

  we can apply induction hypothesis twice, since the number of contractions decreases by 1, while the height might increase. We obtain a derivation of:

  $$\Gamma, \forall x_{\sigma(1)} \dots \forall x_{\sigma(n)} A, \forall x_{\sigma(1)} \dots \forall x_{\sigma(n)} A \vdash \Delta$$

  that we can contract.
- a $\forall$-l rule on $\forall x_1 \dots \forall x_n A$. Then, since π is focused, and since no other rule (but contraction) can apply – remember that axiom, conversion, and weakening rules are atomic – π has the following shape:

  $$\dfrac{\dfrac{\pi_1}{\Gamma, \theta A \vdash \Delta}}{\Gamma, \forall x_1 \dots \forall x_n A \vdash \Delta}\ \forall\text{-l}$$

  where θ is some substitution. Then, it is sufficient to rearrange the quantifications in the order required by the permutation σ, to get π'.
- no other rule can apply to $\forall x_1 \dots \forall x_n A$.

■

So, the order of quantifications and of disjunctions is not important in definition 8, unless we care about derivation size.

6 More on quantification

In this section, we move towards a treatment of quantification and sets of variables (labels) closer to what happens during clausal transformation and resolution steps. Lemma 9 and lemma 11 are extensions to deduction modulo of some miniscoping (or antiprenexing) results [NW01].

As for existential quantification, we deal with Skolemization. Regarding universal quantification, we deal with its scope. This will be important when a free variable is member of only one part of a formula: we eagerly unite sets of labels in definition 8 and we have to be able to specialize universal quantifications again.

6.1 Existential quantifications

Lemma 9 (Specializing existential quantifiers) *Let Γ, Δ be sets of formulas, A be a formula and B a formula where x is not free. Then the sequent $\Gamma, (\exists x A) \vee B \vdash \Delta$ has a derivation if and only if the sequent $\Gamma, \exists x(A \vee B) \vdash \Delta$ has a derivation.*

Proof Given a derivation of $\Gamma, (\exists x A) \vee B \vdash \Delta$, we apply Kleene lemma 3 to get derivations of the sequents:

$$\Gamma, \{c/x\}A \vdash \Delta \qquad \Gamma, B \vdash \Delta$$

c being fresh for the left sequent and chosen fresh for B. We apply first an $\vee$-l rule, and then an $\exists$-l rule to get a derivation of $\Gamma, \exists x(A \vee B) \vdash \Delta$.

Conversely, given a derivation of the former sequent, we apply Kleene lemma 3 to get derivations of the same two sequents as above, that we recombine with first an $\exists$-l and then an $\vee$-l rule. ■

6.2 Universal quantification

Lemma 10 (Pruning) *Let Γ, Δ be sets of formulas, A be a formula and l be its free variables. Let l' be a superset of l. If we have a derivation π of the sequent $\Gamma, \forall^{l'}\mathbf{x}A \vdash \Delta$ then we have a shorter derivation of the sequent $\Gamma, \forall^{l}\mathbf{x}A \vdash \Delta$.*

Proof We refer to definition 8 to explain the notations used here. The proof is performed by induction over π, removing all the unnecessary (not substituting anything) $\forall$-l rules on A: since on every path from the root of π to the leaves (axioms) those rules are present, the height decreases. ■

Lemma 11 (Specializing universal quantifiers) *Let Γ, Δ be sets of formulas, A, B be formulas and I, J be sets of variables. Assume that no y is free in A for any variable $y \in J$. Then we have a derivation π_1 of the sequent:*

$$\Gamma, \forall^I\mathbf{x}\forall^J\mathbf{y}(A \vee B), \ldots, \forall^I\mathbf{x}\forall^J\mathbf{y}(A \vee B) \vdash \Delta$$

if and only if we have a derivation π_2 of the sequent:

$$\Gamma, \forall^I\mathbf{x}(A \vee \forall^J\mathbf{y}B), \ldots, \forall^I\mathbf{x}(A \vee \forall^J\mathbf{y}B) \vdash \Delta$$

Remark 3 In this lemma, we cannot easily control the number of contractions or the height of the derivations, especially for the reverse way. That is why we are considering multiple copies of the formula.

Proof For the direct way, we proceed by induction over the structure of π_1 that we assume focused by lemma 5. If the last rule of π_1 is a rule r on a formula of Γ, Δ, or a contraction on $\forall^I\mathbf{x}\forall^J\mathbf{y}(A \vee B)$, then we apply the induction hypothesis to the derivation(s) of the premise(s), and apply r to the derivation(s) obtained.

Otherwise the last rule must be $\forall$-l on some $\forall^I\mathbf{x}\forall^J\mathbf{y}(A \vee B)$. Since π_1 is focused, it has the following shape, where we omit to mention the variable substitution:

$$\dfrac{\dfrac{\dfrac{\Gamma, A, \ldots, \forall^I \mathbf{x}\forall^J\mathbf{y}(A\vee B) \vdash \Delta \qquad \Gamma, B, \ldots, \forall^I\mathbf{x}\forall^J\mathbf{y}(A\vee B)\vdash\Delta}{\Gamma, A\vee B, \ldots, \forall^I\mathbf{x}\forall^J\mathbf{y}(A\vee B)\vdash\Delta}\,\vee\text{-l}}{\Gamma, \forall^J\mathbf{y}(A\vee B), \ldots, \forall^I\mathbf{x}\forall^J\mathbf{x}(A\vee B)\vdash\Delta}\,\forall\text{-l rules}}{\Gamma,\forall^I\mathbf{x}\forall^J\mathbf{y}(A\vee B),\ldots,\forall^I\mathbf{x}\forall^J\mathbf{x}(A\vee B)\vdash\Delta}\,\forall\text{-l rules}$$

Indeed, no conversion, contraction or axiom rule can apply to $A \vee B$ since it is not atomic. We apply induction hypothesis to the derivation of the premises (also focused), and we rearrange the rules, using the fact that no $y \in J$ is free in A. This gives us the derivation we were looking for:

$$\dfrac{\dfrac{\Gamma, A, \ldots, \forall^I\mathbf{x}(A\vee \forall^J\mathbf{y}B)\vdash\Delta \qquad \dfrac{\Gamma, B, \ldots, \forall^I\mathbf{x}(A\vee\forall^J\mathbf{y}B)\vdash\Delta}{\Gamma,\forall^J\mathbf{y}B,\ldots,\forall^I\mathbf{x}(A\vee\forall^J\mathbf{y}B)\vdash\Delta}\,\forall\text{-l rules}}{\Gamma, A\vee\forall^J\mathbf{y}B,\ldots,\forall^I\mathbf{x}(A\vee\forall^J\mathbf{y}B)\vdash\Delta}\,\vee\text{-l}}{\Gamma,\forall^I\mathbf{x}(A\vee\forall^J\mathbf{y}B),\ldots,\forall^I\mathbf{x}(A\vee\forall^J\mathbf{y}B)\vdash\Delta}\,\forall\text{-l rules}$$

For the converse, we assume π_2 to be focused and we proceed by induction over its structure. If the last rule of π_2 is a rule r on a formula of Γ, Δ, or a contraction on $\forall^I\mathbf{x}(A \vee \forall^J\mathbf{y}B)$ then we apply induction hypothesis to the derivation(s) of the premise(s), and then apply r to the derivation(s) obtained.

Otherwise, it is a $\forall$-l rule on $\forall^I\mathbf{x}(A \vee \forall^J\mathbf{y}B)$ and since π_2 is focused, it has the following shape, where we omit to mention the variable substitution:

$$\dfrac{\dfrac{\Gamma, A, \ldots, \forall^I\mathbf{x}(A\vee \forall^J\mathbf{y}B)\vdash\Delta \qquad \Gamma,\forall^J\mathbf{y}B,\ldots,\forall^I\mathbf{x}(A\vee\forall^J\mathbf{y}B)\vdash\Delta}{\Gamma, A\vee\forall^J\mathbf{y}B,\ldots,\forall^I\mathbf{x}(A\vee\forall^J\mathbf{y}B)\vdash\Delta}\,\vee\text{-left}}{\Gamma,\forall^I\mathbf{x}(A\vee\forall^J\mathbf{y}B),\ldots,\forall^I\mathbf{x}(A\vee\forall^J\mathbf{y}B)\vdash\Delta}\,\forall\text{-l rules}$$

We apply the induction hypothesis to the derivation of both premises, and get two derivations of the sequents $\Gamma', \theta A \vdash \Delta$ and $\Gamma', \forall^J\mathbf{y}B \vdash \Delta$, where Γ' represents the set of formulas $\Gamma, \ldots, \forall^I\mathbf{x}\forall^J\mathbf{y}(A \vee B)$. We use sublemma 12 below, to construct a derivation of the sequent:

$$\Gamma', \Gamma', \forall^J\mathbf{y}(A \vee B) \vdash \Delta, \Delta$$

to which we add $\forall$-l rules and contractions on Γ' and Δ to obtain the desired derivation. ■

Lemma 12 *Let $\Gamma, \Gamma', \Delta, \Delta'$ be sets of formulas. Let C, D be formulas and J a set of variables. Assume that we have derivations of the two sequents:*

$$\Gamma', \forall^J\mathbf{y}D, \ldots, \forall^J\mathbf{y}D \vdash \Delta' \qquad \Gamma, C \vdash \Delta$$

Then we can construct a derivation of the sequent:

$$\Gamma, \Gamma', \forall^J\mathbf{y}(C \vee D), \ldots, \forall^J\mathbf{y}(C \vee D) \vdash \Delta', \Delta$$

Proof By induction over π, the derivation of $\Gamma', \forall^J\mathbf{y}D, \ldots, \forall^J\mathbf{y}D \vdash \Delta'$, assumed by lemma 5 to be focused. We omit the label J. We introduce multiple copies of $\forall\mathbf{y}D$ since the number of contractions is not easily controllable: the derivation of the sequent $\Gamma, C \vdash \Delta$ may contain some and it could be replicated many times in the final derivation.

If the last rule is a rule r on a formula of Γ', Δ' then we apply induction hypothesis, and the same rule r to the derivation(s) obtained. If a fresh constant in π is not fresh for Γ, Δ or C, then first replace it in π by a new fresh constant.

Otherwise, it is a rule on $\forall\mathbf{y}D$. If it is a contraction, we apply induction hypothesis, and then apply contraction to one of the $\forall\mathbf{y}(C \vee D)$ of the conclusion of the obtained derivation. The last possibility is a $\forall$-l rule, and since π is focused, it is:

$$\dfrac{\dfrac{\pi'}{\Gamma', D, \forall \mathbf{y} D, \ldots, \forall \mathbf{y} D \vdash \Delta'}\text{ rule on } D}{\Gamma', \forall \mathbf{y} D, \forall \mathbf{y} D, \ldots, \forall \mathbf{y} D \vdash \Delta'}\ \forall\text{-l rules}$$

This is also valid if J is empty. We apply induction hypothesis to π' and we obtain a derivation of the sequent:

$$\Gamma, \Gamma', D, \forall \mathbf{y}(C \vee D), \ldots, \forall \mathbf{y}(C \vee D) \vdash \Delta, \Delta'$$

then, we can construct the following derivation, using the derived weakening rule:

$$\dfrac{\dfrac{\dfrac{\Gamma, C \vdash \Delta}{\Gamma, \Gamma', C, \ldots, \forall \mathbf{y}(C \vee D) \vdash \Delta, \Delta'}\text{ weak} \qquad \dfrac{\pi'}{\Gamma, \Gamma', D, \ldots, \forall \mathbf{y}(C \vee D) \vdash \Delta, \Delta'}}{\Gamma, \Gamma', C \vee D, \ldots, \forall \mathbf{y}(C \vee D) \vdash \Delta, \Delta'}\ \vee\text{-l}}{\Gamma, \Gamma', \forall \mathbf{y}(C \vee D), \ldots, \forall \mathbf{y}(C \vee D) \vdash \Delta, \Delta'}\ \forall\text{-l rules}$$

■

7 Clause normal form transformation in the sequent calculus

7.1 The Skolem theorem in sequent calculus modulo

Calculating the clause form performs Skolemization on the fly, so we must be able to simulate this in the sequent calculus. Skolem theorem is well-known [Sko23] in many frameworks, however none of the existing results apply to our deduction modulo case. There is only one known proof [Her08], but it is semantic, and translating it back to syntax would introduce cuts. Since our derivations are forced cut-free, we must additionally ensure that no cut rule are introduced and carry out a syntactic proof. We therefore adapt the proof of [Mae55], reworked by [Min66, DW05b], to our case.

This gives a result of independent interest: a Skolem theorem for (classical) deduction modulo theories, and therefore for all the theories one can express with it.

Theorem 3 (Skolem theorem) *Let Γ, Δ be sets formulas, A be a formula, $x_1, \ldots, x_n$ be n variables abbreviated as* $\mathbf{x}$ *and f be a n-ary function symbol, fresh with respect to Γ, Δ, A* and *rewrite rules.*

There is a derivation of the sequent $\Gamma, \forall \mathbf{x} \exists y A \vdash \Delta$ if and only if there is a derivation of the sequent $\Gamma, \forall \mathbf{x}\{f(\mathbf{x})/y\}A \vdash \Delta$

The *only if* part has a trivial proof since, when applying the $\exists$-l rule, we introduce a fresh constant c, that lemma 2 allows to replace by any term in the derivation of the resulting statement. Here, we need a proof of the converse, and since we are in theorem 3 also interested in the case where the cut-elimination property *fails*, we cannot rely on cut elimination.

The rest of this section is dedicated to the proof of the converse statement. A, f and the variables $\mathbf{x}$ are fixed. Until the end of this section, we consider a restriction of sequent calculus modulo (figure 1) where sequents are always closed, and therefore instantiations in rules $\forall$-l and $\exists$-r are limited to ground terms. The calculus of figure 1 is conservative over it: given a derivation π of the calculus of figure 1 we can replace (extending definition 4) each free variable x by a new fresh constant c_x, the resulting derivation is still a valid derivation and it fits our new constraint. Therefore, if a sequent $\Gamma \vdash \Delta$ is closed, it has a derivation in the calculus of figure 1 if and only if it has a derivation in the restricted calculus.

We do not impose this restriction in the whole article since it would complicate the study of the relations between sequent calculus and resolution (that allows free variables) in section 8. Here, on the contrary, we consider only ground instantiations and closed sequents as this simplifies definitions and lemmas, compared to [Mae55, DW05b] for instance.

Definition 9 ([DW05b]) Let $\mathbf{t} = t_1, ..., t_n$ be ground terms, and $\mathbf{x} = x_1, ..., x_n$ be distinct variables.

- $\{\mathbf{t}/\mathbf{x}\}$ denotes the parallel substitution $\{t_1/x_1, ..., t_n/x_n\}$[1]
- an *f-term* is a term of the form $f(\mathbf{t})$

Remark 4 We do not need the notion of *partial instance* [DW05b] (or f-formula [Mae55]). Those intermediate states of the formula $\forall\mathbf{x}\{f(\mathbf{x})/y\}A$, where not every $\forall$-l rule has been applied yet, will be handled by focusing instead. This greatly simplifies the matter.

Definition 10 (Pruning f-terms) Let u be a term, let $f(\mathbf{t}_1), ..., f(\mathbf{t}_k)$ be the f-terms appearing in u, arranged by decreasing term size, so that $f(\mathbf{t}_j)$ does not appear in $f(\mathbf{t}_i)$ whenever $j < i$. Let $c_1, ..., c_k$ be k fresh constants. We define the sequence of terms u^i:

- $u^0 = t$,
- u^{i+1} is u^i where each occurrence of $f(\mathbf{t_i})$ has been replaced by c_i.

We let u^* be u^k. We extend this definition to $\mathbf{u}^*$ for a list of terms $\mathbf{u}$, to A^* for a formula A and to Γ^* for a context Γ.

Example 1 $A(f(c), f(f(c)), g(f(c)))^*$ is $A(c_2, c_1, g(c_2))$, and not $A(c_1, f(c_1), g(c_1))$. This is the whole point: we prune the outermost f-terms first.

Proposition 2 *Let Γ, Δ be sets of (closed) formulas where all f-terms are ground and let Σ denote an arbitrary number of copies of $\forall\mathbf{x}\{f(\mathbf{x})/y\}A$. Let $f(\mathbf{t}_1), \dots, f(\mathbf{t}_p)$ the f-terms appearing in Γ, Δ and let Θ be the set of formulas $\{\mathbf{t}_1^*/\mathbf{x}, c_1/y\}A, ..., \{\mathbf{t}_p^*/\mathbf{x}, c_p/y\}A$ (relevant instances of A).*

If we have a derivation of $\Gamma, \Sigma \vdash \Delta$ then we can build a derivation of $\Gamma^*, \Theta, \forall\mathbf{x}\exists yA \vdash \Delta^*$.

Proof By induction on the derivation π of $\Gamma, \Sigma \vdash \Delta$ which we assume to be focused.

- if the last rule is an axiom, it cannot involve a formula of Σ since all f-terms are ground in Δ. Therefore, it is between a formula of Γ and a formula of Δ. We replace it by an axiom between the corresponding formulas of Γ^* and Δ^*.
- if the last rule is a rule on $\forall\mathbf{x}\{f(\mathbf{x})/y\}A \in \Sigma$, which is:
  - a structural rule. Induction hypothesis gives us directly the wanted derivation.
  - a $\forall$-l rule. Since the derivation is focused, we have:

$$\frac{\Gamma, \Sigma, \{\mathbf{t}/\mathbf{x}, f(\mathbf{t})/y\}A \vdash \Delta}{\Gamma, \Sigma, \forall\mathbf{x}\{f(\mathbf{x})/y\}A \vdash \Delta} \ \forall\text{-l rules}$$

    Note that f does not appear in A by hypothesis. So $(\{\mathbf{t}/\mathbf{x}, f(\mathbf{t})/y\}A)^* = \{\mathbf{t}^*/\mathbf{x}, c/y\}A$. By induction hypothesis, we have then a derivation of the sequent:

$$\Gamma^*, \{\mathbf{t}^*/\mathbf{x}, c/y\}A, \Theta, \Theta', \forall x\exists yA \vdash \Delta^*$$

    where Θ' is the set $\left\{\{\mathbf{u}_1^*/\mathbf{x}, d_1/y\}A, ..., \{\mathbf{u}_q^*/\mathbf{x}, d_q/y\}A\right\}$, representing all the instances $\{\mathbf{u}_i^*/\mathbf{x}, d_i/y\}A$ such that $f(\mathbf{u}_i)$ appears in $f(\mathbf{t})$ and not in Γ, Δ.

[1] identical to $\{t_1/x_1\}...\{t_n/x_n\}$ since in this section, terms are ground.

Assume, similarly to definition 10, that $\mathbf{u}_1, ..., \mathbf{u}_q$ are arranged by increasing size, so that $d_j = (f(\mathbf{u}_j))^*$ does not appear in $\mathbf{u}_i^*$ if $j > i$. With this definition, $\{\mathbf{u}_q^*/\mathbf{x}, d_q/y\}A = \{\mathbf{t}^*/\mathbf{x}, c/y\}A$ and we have, by hypothesis and contraction, a derivation of the sequent:

$$\Gamma^*, \Theta, \Theta', \forall \mathbf{x} \exists y A \vdash \Delta^*$$

We prove by induction on q that we can get rid of Θ'. If Θ' is empty, no operation is required, note only that in this case $\{\mathbf{t}^*/\mathbf{x}, c/y\}A \in \Theta$ so that we still have a valid derivation of the sequent $\Gamma^*, \Theta, \Theta', \forall \mathbf{x} \exists y A \vdash \Delta^*$.
For the inductive case, let $\Theta'' = \Theta' \setminus \{\mathbf{u}_q^*/\mathbf{x}, d_q/y\}A$. From our ordering on the elements of Θ', d_q is fresh in $\{\mathbf{u}_q^*/\mathbf{x}, d_q/y\}A$, and the following derivation proves the inductive case:

$$\cfrac{\cfrac{\cfrac{\Gamma^*, \Theta, \Theta'', \{\mathbf{u}_q^*/\mathbf{x}, d_q/y\}A, \forall \mathbf{x} \exists y A \vdash \Delta^*}{\Gamma^*, \Theta, \Theta'', \{\mathbf{u}_q^*/\mathbf{x}\}\exists y A, \forall \mathbf{x} \exists y A \vdash \Delta^*}\ \exists\text{-l}}{\Gamma^*, \Theta, \Theta'', \forall \mathbf{x} \exists y A, \forall \mathbf{x} \exists y A \vdash \Delta^*}\ \forall\text{-l rules}}{\Gamma^*, \Theta, \Theta'', \forall \mathbf{x} \exists y A \vdash \Delta^*}\ \text{contr-l}$$

- otherwise we apply the same rule. For instance:
  - if it is a $\forall$-l rule,

$$\frac{\Gamma, \Sigma, \{u/z\}B \vdash \Delta}{\Gamma, \Sigma, \forall z B \vdash \Delta}\ \forall\text{-l}$$

  Since all the f-terms in B are ground, $(\{u/z\}B)^* = \{u^*/z\}B^*$ and $(\forall z B)^* = \forall z B^*$. We can build the derivation:

$$\frac{\Gamma^*, \{u^*/z\}B^*, \Theta, \Theta', \forall x \exists y A \vdash \Delta^*}{\Gamma^*, \forall z B^*, \Theta, \Theta', \forall x \exists y A \vdash \Delta^*}\ \forall\text{-l}$$

  where the upper sequent has been proved by induction hypothesis and, as in the previous case, Θ' contains all the instances $\{\mathbf{u}_i^*/\mathbf{x}, d_i/y\}A$ such that $f(\mathbf{u}_i)$ appears in u and not in $\Gamma, \forall z B, \Delta$. With the exact same method, we get rid one by one of those instances, by taking care of the freshness of the d_i in ordering them.
  - if it is a $\exists$-l rule,

$$\frac{\Gamma, \Sigma, \{d/z\}B \vdash \Delta}{\Gamma, \Sigma, \exists z B \vdash \Delta}\ \forall\text{-l}$$

  By induction hypothesis, we have a derivation of the sequent:

$$\Gamma^*, (\{d/z\}B)^*, \Theta \vdash \Delta^*$$

  and since in B all f-term are ground, $(\{d/z\}B)^* = \{d/z\}(B^*)$. d does not appear in any f-term, and therefore does not appear in Γ^*, Δ^* or Θ, so we can safely apply the $\exists$-l rule to this sequent as well.
  - The $\exists$-r case is similar to the $\forall$-l case. The other logical rules do not present any difficulty, they only may involve some (derived) weakening on Θ. The only remaining case is that of a rewrite rule, which is also straightforward, since no f-term can be introduced thanks to the hypothesis that f does not appear in the rewrite system. ■

Now we can go back to Skolem theorem 3.

Proof We apply proposition 2: f is fresh, so $\Gamma^* = \Gamma$, $\Delta^* = \Delta$ and Θ is empty. ■

The following lemma says that we can strengthen a formula in a disjunction appearing as an hypothesis, without changing the provability of the sequent.

Lemma 13 (Strengthening hypothesis) *Let Γ, Δ be sets of formulas, and let A, B, C be formulas. If we have a derivation π of a sequent $\Gamma, \forall^l \mathbf{x}(A \vee C) \vdash \Delta$, then we can construct a derivation π' of the sequent $\Gamma, \forall^l \mathbf{x}((A \wedge B) \vee C) \vdash \Delta$ containing the same number of contractions as π.*

Proof By induction over the pair $\langle C(\pi), h(\pi)\rangle$. If the last rule of π is a rule r on a formula of Γ, Δ or a $\forall$-l rule on $\forall^l \mathbf{x}(A \vee C)$, we apply induction hypothesis to the premise(s) and add r to the obtained derivation.

If the last rule is a contraction on $\forall^l \mathbf{x}(A \vee C)$ we have a derivation of the sequent $\Gamma, \forall^l \mathbf{x}(A \vee C), \forall^l \mathbf{x}(A \vee C) \vdash \Delta$. An application of induction hypothesis gives us a derivation of the sequent:

$$\Gamma, \forall^l \mathbf{x}((A \wedge B) \vee C), \forall^l \mathbf{x}(A \vee C) \vdash \Delta$$

containing one less contraction. So we apply again induction hypothesis and contract the obtained derivation.

Lastly, if the last rule is $\vee$-left (l is empty), we have a derivation of the sequents $\Gamma, A \vdash \Delta$ and $\Gamma, B \vdash \Delta$ and we build the following derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma, A \vdash \Delta}{\Gamma, A, B \vdash \Delta}\ \text{derived weak-l}}{\Gamma, A \wedge B \vdash \Delta}\ \wedge\text{-l} \qquad \Gamma, C \vdash \Delta}{\Gamma, (A \wedge B) \vee C \vdash \Delta}\ \vee\text{-l}$$

Note that the derived rule weak-l does not introduce contractions. ■

Proposition 3 (Inversion of clausal transformation) *Let $\psi_1, \dots, \psi_n, \chi_1, \dots, \chi_m$ be sets of labelled formulas such that from a clausal transformation of figure 3 $\psi_1 \mid \dots \mid \psi_n \rightsquigarrow \chi_1 \mid \dots \mid \chi_m$.*
If we have a derivation of the sequent $\overline{\chi_1}, \dots, \overline{\chi_m} \vdash$ then we can construct a derivation of the sequent $\overline{\psi_1}, \dots, \overline{\psi_n} \vdash$.

If $\psi_1 \mid \dots \mid \psi_n \rightsquigarrow^* \chi_1 \mid \dots \mid \chi_m$ using an arbitrary number of clausal-transformation steps, the same result holds.

Proof The second part of the proposition is the transitive closure of the first, on which we concentrate. We omit the labels when they are not relevant, and we will use indifferently $\overline{\psi}$ or $\forall^l \mathbf{x} \bigvee \psi$. According to lemma 7, if ψ is empty, $A \vee \bigvee \psi$ is equivalently A or $A \vee \bot$. We consider each rule of figure 3, assumed without loss of generality to apply to ψ_1:

- $\Phi \mid \psi, (A \wedge B)^l \rightsquigarrow \Phi \mid \psi, A^l \mid \psi, B^l$. We have by hypothesis a derivation π of the sequent:

  $$\overline{\psi, A}, \overline{\psi, B}, \overline{\psi_2}, \dots, \overline{\psi_n} \vdash$$

  By lemma 6 we can assume A and B to be at first place in the disjunctions $\overline{\psi, A}$ and $\overline{\psi, B}$. We use lemma 13 twice to get a derivation of:

  $$\overline{(A \wedge B), \psi}, \overline{(A \wedge B), \psi}, \overline{\psi_2}, \dots, \overline{\psi_n} \vdash$$

  that we contract. Note that labels for quantifications do not change.

The case of the dual rule $\Phi \mid \psi, (\neg(A \vee B))^l \rightsquigarrow \Phi \mid \psi, (\neg A)^l \mid \psi, (\neg B)^l$ is handled in exactly the same way, except that we would have to apply a lemma similar to lemma 13, stating that $\neg A$ can be strengthened in $\neg(A \vee B)$ and using Kleene lemma 3.

- $\Phi \mid \psi, (A \vee B)^l \rightsquigarrow \Phi \mid \psi, A^l, B^l$. No operation is required since $\overline{\psi, (A \vee B)^l}$ is the same as $\overline{\psi, A^l, B^l}$, modulo lemma 6. The case of the dual rule is almost as simple, since Kleene lemma 3 shows that anywhere in any derivation, $\neg(A \wedge B)$ could be replaced safely by $\neg A \vee \neg B$ and conversely.
- $\Phi \mid \psi, (A \Rightarrow B)^l \rightsquigarrow \Phi \mid \psi, (\neg A)^l, B^l$ is similar to the $\vee$ case, while $\Phi \mid \psi, (\neg(A \Rightarrow B))^l \rightsquigarrow \Phi \mid \psi, A^l \mid \psi, (\neg B)^l$ is similar to the $\wedge$ case.
- $\Phi \mid \psi, \bot^l \rightsquigarrow \Phi \mid \psi$. By a straightforward induction using lemma 7 to replace $\bigvee \psi_1$ by $(\bigvee \psi_1) \vee \bot$, we transform a derivation of $\overline{\psi_1}, \dots, \overline{\psi_n} \vdash$ into a derivation of $\overline{\psi_1, \bot}, \dots, \overline{\psi_n} \vdash$.
- the case of the rule $\Phi \mid \psi, (\neg\bot)^l \rightsquigarrow \Phi$ is even more straightforward. Just add a derived weakening rule on $\overline{\psi_1, (\neg\bot)}$.
- $\Phi \mid \psi, (\forall x A)^{\mathbf{y}} \rightsquigarrow \Phi \mid \psi, A^{\mathbf{y},x}$. x being fresh from the clausal transformation rule (figure 3), if the formula $\overline{\psi_1, (\forall x A)^{\mathbf{y}}}$ is $\forall^l \mathbf{z}((\bigvee \psi_1) \vee \forall x A)$, then the formula $\overline{\psi_1, A^{x,\mathbf{y}}}$ is $\forall^{x,l}\, \mathbf{z}((\bigvee \psi_1) \vee A)$. We apply lemma 11 to the derivation of $\overline{\psi_1, A^{x,\mathbf{y}}}, \overline{\psi_2}, \dots, \overline{\psi_n} \vdash$ and this gives us a derivation of the needed sequent. The case of the dual rule is similar, as usual using Kleene lemma 3.
- $\Phi \mid \psi, (\exists x A)^{\mathbf{y}} \rightsquigarrow \Phi \mid \psi, (\{f(\mathbf{y})/x\}A)^{\mathbf{y}}$. Assume that we have a derivation of the sequent:

$$\forall^l \mathbf{z}\Big(\{f(\mathbf{y})/x\}A \vee \bigvee \psi_1\Big), \overline{\psi_2}, \dots, \overline{\psi_n} \vdash$$

Suppose, without loss of generality, that x is fresh, otherwise letting x' fresh, first replace x by x' in A. We permute the quantifiers by lemma 8 and get a derivation of the sequent:

$$\forall^{\mathbf{y}} \mathbf{y} \forall^{l \setminus \mathbf{y}} \mathbf{z}\, \{f(\mathbf{y})/x\}\Big(A \vee \bigvee \psi_1\Big), \overline{\psi_2}, \dots, \overline{\psi_n} \vdash$$

Thanks to lemma 11, since $\mathbf{y}$ is a superset of the free variables of $\{f(\mathbf{y})/x\}A$ and x does not appear in ψ_1, we can transform this derivation into a derivation of the sequent:

$$\forall^{\mathbf{y}} \mathbf{y}\, \{f(\mathbf{y})/x\}\Big(A \vee \forall^{l \setminus \mathbf{y}} \mathbf{z} \bigvee \psi_1\Big), \overline{\psi_2}, \dots, \overline{\psi_n} \vdash$$

We now use Skolem theorem 3 to get a derivation of the sequent:

$$\forall^{\mathbf{y}} \mathbf{y} \exists x (A \vee \forall^{l \setminus \mathbf{y}} \mathbf{z} \bigvee \psi_1), \overline{\psi_2}, \dots, \overline{\psi_n} \vdash$$

Afterwards, we use lemma 9 to get a derivation of the sequent:

$$\forall^{\mathbf{y}} \mathbf{y}((\exists x A) \vee \forall^{l \setminus \mathbf{y}} \mathbf{z} \bigvee \psi_1), \overline{\psi_2}, \dots, \overline{\psi_n} \vdash$$

Then, we apply lemma 11 in the opposite way to get back the quantifiers in front, and get a derivation of the sequent:

$$\forall^l \mathbf{z}((\exists x A) \vee \bigvee \psi_1), \overline{\psi_2}, \dots, \overline{\psi_n} \vdash$$

The case of the dual rule is similar and uses Kleene lemma 3 to show the equivalence of $\neg\forall$ and $\exists\neg$.

- $\Phi \mid \psi, (\neg\neg A)^l \rightsquigarrow \Phi \mid \psi, A^l$ is an easy induction using lemma 3 twice. ■

8 Resolution in sequent calculus

Now that we know that the clausal form transformation rules of figure 3 are admissible in the (cut free) sequent calculus of figure 1, let us see how we can emulate the EIR rules of figure 4. We begin with a backward encoding of the identical resolution rule. Lemma 7 allows to handle clause containing only one literal (i.e. $A \vee B$ below has in fact the shape B).

Lemma 14 (Identical Resolution) *Let Γ, Δ be sets of formulas, A, B, C be formulas and l be a set of variables. If we have a derivation π of the sequent $\Gamma, \forall^l \mathbf{x}(A \vee B), \dots, \forall^l \mathbf{x}(A \vee B) \vdash \Delta$, then we can build a derivation of the sequent:*

$$\Gamma, \forall^l \mathbf{x}(A \vee C), \forall^l \mathbf{x}(B \vee \neg C), \dots, \forall^l \mathbf{x}(A \vee C), \forall^l \mathbf{x}(B \vee \neg C) \vdash \Delta$$

Proof Like in lemma 11, we introduce multiple copies. Quantifying three times over the same l is slightly too weak to reflect the Identical Resolution rule (figure 4), which is treated in proposition 4. The proof proceeds by induction on π, that we assume focused.

If the last rule of π is a rule r on a formula of Γ, Δ or a contraction on $\forall^l \mathbf{x}(A \vee B)$, we apply induction hypothesis, and then the same rule r (*twice* in the second case). Otherwise, it is a $\forall$-l rule on $\forall^l \mathbf{x}(A \vee B)$. Since π is focused, it has the shape:

$$\dfrac{\dfrac{\Gamma, \theta A, \dots, \forall^l \mathbf{x}(A \vee B) \vdash \Delta \qquad \Gamma, \theta B, \dots, \forall^l \mathbf{x}(A \vee B) \vdash \Delta}{\Gamma, \theta(A \vee B), \dots, \forall^l \mathbf{x}(A \vee B) \vdash \Delta}\ \vee\text{-l}}{\Gamma, \forall^l \mathbf{x}(A \vee B), \dots, \forall^l \mathbf{x}(A \vee B) \vdash \Delta}\ \forall\text{-l rules}$$

where θ is the instantiation of the variables of l. Applying induction hypothesis on the premises and letting Γ' be the set of formulas $\Gamma, \forall^l \mathbf{x}(A \vee C), \forall^l \mathbf{x}(B \vee \neg C) \dots, \forall^l \mathbf{x}(A \vee C), \forall^l \mathbf{x}(B \vee \neg C)$, we get derivations of the sequents $\Gamma', \theta A \vdash \Delta$ and $\Gamma', \theta B \vdash \Delta$. We conclude by building the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma', \theta A \vdash \Delta}{\Gamma', \theta A, \theta\neg C \vdash \Delta}\ \text{weak} \qquad \dfrac{\dfrac{}{\Gamma', \theta C \vdash \theta C, \Delta}\ \text{axiom}}{\Gamma', \theta C, \theta\neg C \vdash \Delta}\ \neg\text{-l}}{\Gamma', \theta(A \vee C), \theta\neg C \vdash \Delta}\ \vee\text{-l} \qquad \dfrac{\Gamma', \theta B \vdash \Delta}{\Gamma', \theta(A \vee C), \theta B \vdash \Delta}\ \text{weak}}{\Gamma', \theta(A \vee C), \theta(B \vee \neg C) \vdash \Delta}\ \vee\text{-l}}{\Gamma', \forall^l \mathbf{x}(A \vee C), \forall^l \mathbf{x}(B \vee \neg C) \vdash \Delta}\ \forall\text{-l rules}$$

where, as we can note, the resolution step between $A \vee C$ and $B \vee \neg C$ turns into an axiom with C. ■

Lemma 15 (Instantiation) *Let Γ, Δ be sets of formulas, A be a formula. Let l, l' be sets of variables, t be a term which free variables are in l', and z be a fresh variable. Assume that no free variable of A is in $l' \backslash l$. If we have a derivation π of the sequent $\Gamma, \forall^{l' \cup l} \mathbf{x}\{t/z\}A \vdash \Delta$ then we can build a derivation of the sequent $\Gamma, \forall^{z,l} \mathbf{x}A \vdash \Delta$ having the same number of contractions as π.*

Remark 5 We cannot assume the intersection $l \cap l'$ to be empty, since the Instantiation rule of figure 4 does not impose anything.

Proof By induction over the pair $\langle C(\pi), h(\pi)\rangle$. If $l \cup l'$ is empty it is sufficient to add a $\forall$-l rule to π, so we assume that $l \cup l'$ is not empty.

If the last rule of π is a rule r on Γ, Δ, apply induction hypothesis to the premise(s) and then the same rule r. Otherwise the last rule is a rule on $\forall^{l' \cup l} \mathbf{x}\{t/z\}A$.

If it is a contraction, then apply induction hypothesis twice – as usual the decreased number of contraction allows that – and then a contraction on $\forall^{z,l}\mathbf{x}A_1$.

Otherwise it is a ∀-l rule on a variable y, since $l' \cup l$ is not empty. We apply induction hypothesis, get a derivation, and identify two cases:

- $y \in l$. We add a ∀-l rule on y to the derivation, and we use lemma 8 to permute y and z in the quantification order. This does not add contraction rules.
- $y \in l' \backslash l$. Induction hypothesis directly gives us $\forall^{z,l}\mathbf{x}A$, since y is not free in A. ■

Proposition 4 (EIR rules) *Let $\psi_1, \ldots, \psi_n$ be clauses. If $\psi_1 \ldots, \psi_n \hookrightarrow_{\mathcal{RE}} \square$ then we can build a derivation of $\overline{\psi_1}, \ldots, \overline{\psi_n} \vdash$.*

Proof By induction over the length of the resolution derivation. If it has no steps, one of the clauses, say ψ_1, is empty. Then $\overline{\psi_1} = \forall^l\mathbf{x}\bot$. Some ∀-l rules and a ⊥-l rule allow us to conclude. Otherwise, the last rule of the derivation is:

- Identical Resolution. Assume without loss of generality that this is on the clauses $\psi_1 = \psi_1', C$ and $\psi_2 = \psi_2', \neg C$. Applying induction hypothesis and lemma 6, we get a derivation of the sequent $\forall^l\mathbf{x}((\bigvee \psi_1') \vee (\bigvee \psi_2')), \overline{\psi_3}, \ldots, \overline{\psi_n} \vdash$. Applying lemma 14, we get a derivation of the sequent $\forall^l\mathbf{x}(C \vee \bigvee \psi_1'), \forall^l\mathbf{x}(\neg C \vee \bigvee \psi_2'), \overline{\psi_3}, \ldots, \overline{\psi_n} \vdash$. Note that the formula $\forall^l\mathbf{x}(\bigvee\{C, \psi_1'\})$ is not exactly $\overline{\psi_1', C}$, because of the label l. l could contain variables not in the labels of ψ_1', C and it could also miss some variables in the label of C. So we add the required ∀-l rules and prune the unnecessary variables, by lemma 10.
- Reduction. Assume that it applies to $\psi_1 \longrightarrow \phi$, where ϕ is a set of formulas (no longer a clause). Let $\psi \in cl(\phi) = \{\chi_1, \ldots, \chi_m\}$ the clause such that we have a shorter derivation of $\psi_1, \ldots, \psi_n, \psi \hookrightarrow_{\mathcal{RE}} \square$. Then, by induction hypothesis, we have a derivation of the sequent $\overline{\psi_1}, \ldots, \overline{\psi_n}, \overline{\psi} \vdash$. We can apply the derived weakening rule to get a derivation of the sequent:
$$\overline{\psi_1}, \ldots, \overline{\psi_n}, \overline{\chi_1}, \ldots, \overline{\chi_m} \vdash$$
Applying proposition 3 gives us a derivation of the sequent:
$$\overline{\psi_1}, \ldots, \overline{\psi_n}, \overline{\phi} \vdash$$
We have $\overline{\psi_1} \longrightarrow \overline{\phi}$, since this relation holds for ψ_1 and ϕ. So we apply a conv-l rule:
$$\frac{\overline{\psi_1}, \ldots, \overline{\psi_n}, \overline{\phi} \vdash}{\overline{\psi_1}, \ldots, \overline{\psi_n}, \overline{\psi_1} \vdash} \text{ derived conv-l } \overline{\psi_1} \longrightarrow \overline{\phi}$$
and then we contract on $\overline{\psi_1}$.
- Conversion, on $\psi_1 \equiv_{\mathcal{E}} \psi$. Note that ψ remains a clause since conversion acts only on terms. Induction hypothesis gives us a derivation of the sequent $\overline{\psi}, \overline{\psi_1}, \ldots, \overline{\psi_n} \vdash$. Since $\psi_1 \longrightarrow^* \psi$ (definition 2) we can add a derived conv-l rule and then contract.
- Instantiation, on ψ_1, of x by t. Induction hypothesis gives us a derivation of the sequent $\overline{\{t/x\}\psi_1}, \overline{\psi_1}, \ldots \overline{\psi_n} \vdash$. We apply lemma 15, l being the union of the labels of ψ_1 without x, and l' the free variables of t. This gives a derivation of $\overline{\psi_1}, \overline{\psi_1}, \ldots, \overline{\psi_n} \vdash$, which we contract.

Finally, if we did not obtain the same quantification order, we apply lemma 8. ■

We state again theorem 2:

Theorem 4 (Soundness of EIR) *Let Γ, Δ be sets of formulas. If $cl(\Gamma, \neg\Delta) \hookrightarrow_{\mathcal{RE}} \square$ then we can build a derivation of the sequent $\Gamma \vdash \Delta$.*

Proof Let $\psi_1, \dots, \psi_n = cl(\Gamma, \neg\Delta)$ and assume $\psi_1, \dots, \psi_n \hookrightarrow_{\mathcal{RE}} \Box$. Proposition 4 gives us a derivation of the sequent $\overline{\psi_1}, \dots, \overline{\psi_n} \vdash$. We apply proposition 3 to get a derivation of the sequent $\Gamma, \neg\Delta \vdash$ and then Kleene lemma 3 to get a derivation of the sequent $\Gamma \vdash \Delta$. ■

9 Conclusion

In this article, we proved that a resolution proof corresponds to a cut-free sequent-calculus derivation in the framework of deduction modulo, without *any* hypothesis on the rewrite system used. However, if one wants completeness of the original ENAR with respect to the original sequent calculus, one has to make the extra assumption that the rewrite system is confluent, to be able to use proposition 1.

This applies to many interesting cases, including higher-order logic [DW03,DHK01], Peano's arithmetic [DW05a], or Zermelo's set theory [DM07].

Moreover, since the cut-free fragment of sequent calculus corresponds exactly to the resolution method, the completeness theorem cannot be proved if we do not prove (or use, as in [DHK03]) some version of the cut-elimination theorem. In particular the semantic completeness result of [Stu01], combined with theorem 2 proved here and soundness of sequent-calculus modulo with cuts, entails directly a semantic cut-elimination theorem. More generally, any resolution completeness theorem not using a cut-free derivation, combined with theorem 2, entails a cut-elimination theorem for that given theory.

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