Generalizing Boolean Algebras for Deduction Modulo

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Credits: all the people from INRIA ARC CORIAS

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Deduction modulo [Dowek, Hardin & Kirchner]

Original idea: *combine automated theorem proving with rewriting*

Generalized to: *combine any deduction process with rewriting*

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Example: Classical Sequent Calculus Modulo

$$\text{LK} \quad + \quad \frac{\Gamma \vdash A, \Delta \qquad A \equiv B}{\Gamma \vdash B, \Delta}\ \textsc{Conversion Right} \quad + \quad \frac{\Gamma, A \vdash \Delta \qquad A \equiv B}{\Gamma, B \vdash \Delta}\ \textsc{Conversion Left}$$

Examples of theories expressed in Deduction Modulo

- arithmetic
- simple type theory (HOL)
- confluent, terminating and quantifier free rewrite systems
- confluent, terminating and positive rewrite systems
- positive rewrite system such that each atomic formula has at most one one-step reduct

What about cut-elimination ?

$$\begin{cases} \vdash \text{even}(0) \\ \text{even}(n) \vdash \text{even}(n+2) \end{cases}$$

$$\text{Cut}\ \frac{\vdash \text{even}(0) \qquad \text{even}(0) \vdash \text{even}(2)}{\vdash \text{even}(2)}$$

What about cut-elimination ?

$$\begin{cases} \text{even}(0) & \rightarrow & \top \\ \text{even}(x+2) & \rightarrow & \text{even}(x) \end{cases}$$

$$\dfrac{\dfrac{}{\vdash \top} \qquad \text{even}(2) \equiv \top}{\vdash \text{even}(2)}$$

Cut-elimination implies consistency… and we must pay the prize

Consistency $\Leftarrow$ $\not\Rightarrow$ Cut elimination $\Leftarrow$ $\not\Rightarrow$ Normalization

minimal counterexample : $A \rightarrow A \Rightarrow B$

convergent counterexample : $\left\{ \begin{array}{lcl} R \in R & \rightarrow & \forall y. y \simeq R \Rightarrow y \in R \Rightarrow B \\ y \simeq z & \rightarrow & \forall y. (x \in y \Rightarrow z \in y) \end{array} \right.$

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convergent counterexample : $\left\{ \begin{array}{lcl} R \in R & \rightarrow & \forall y. y \simeq R \Rightarrow y \in R \Rightarrow A \Rightarrow A \\ y \simeq z & \rightarrow & \forall y. (x \in y \Rightarrow z \in y) \end{array} \right.$

Superconsistency (SC): A generic criterion

Dowek & Werner: *Proof normalization modulo*
Dowek: *Truth values algebras and proof normalization*

Consistency

A theory $\mathcal{T}$ is consistent if it can be interpreted in **one** model not reduced to $\bot$

Super-consistency

A theory $\mathcal{T}$ is super-consistent if it can be interpreted in **all** models

What is the notion of model ?

Pre-Heyting Algebras

…are Heyting algebras generalized to *pre-ordered sets*

Pre-Heyting algebras take into account two distinct notion of equivalence:

Computational equivalence : strong, corresponds to equality in the model

Logical equivalence : loose corresponds to $\geqslant \cap \leqslant$

Superconsistency (SC): characterizing analytical theories

Dowek's remark

The set of reducibility candidates for NJ modulo is a pre-Heyting Algebra.

Superconsistency (SC): characterizing analytical theories

Dowek's remark

The set of reducibility candidates for NJ modulo is a pre-Heyting Algebra.

Consistency The theory can be interpreted in a non-trivial model

Superconsistency The theory can be interpreted in any model

Any superconsistent theory can then be interpreted in the pre-Heyting algebra of reducibility candidates.

Conclusion

Any superconsistent theory is strongly normalizable (for NJ)

Examples of theories proved to be superconsistent

- arithmetic
- simple type theory
- confluent, terminating and quantifier free rewrite systems
- confluent, terminating and positive rewrite systems
- positive rewrite system such that each atomic formula has at most one one-step reduct

Now what about classical sequent calculi ?

- the framework:
  - monosided classical sequent calculus
  - deduction modulo with explicit conversion
  - negation is an operation and not a connective
- the aim: *direct* proof that SC implies cut elimination in $LK_{\equiv}$
- the method: sequent reducibility candidates [Dowek, Hermant].

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- the framework:
  - ★ monosided classical sequent calculus
  - ★ deduction modulo with explicit conversion
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Pre-Boolean Algebras

- similar as for Heyting's: weaken the order in Boolean Algebras into a pre-order (*i.e.* loose antisymmetry)
- but stricter: $a^{\perp\perp} = a$ (and not $a^{\perp\perp} \leq\geq a$)

A road map/recipe

Suppose you have an unspecified superconsistent theory

Step 1 Construct a set of reducibility candidates

Step 2 Prove that it is a pre-Boolean algebra

you get an interpretation of sequents in the algebra for free thanks to superconsistency

Step 3 Prove adequacy: provable sequents are in their interpretations

you get cut-elimination as a direct corollary

Inheritance from Linear Logic [Okada, Brunel]

- identifying a site in sequents: pointed sequents

$$\vdash \Delta, A^\circ$$

- interaction: a partial function ★

$$\vdash \Delta_1, A^\circ \star \vdash \Delta_2, B^\circ = \vdash \Delta_1, \Delta_2 \qquad \text{if } A \equiv B^\perp$$

$$\vdash \Delta_1, A^\circ \star X = \{ \vdash \Delta_1, \Delta_2 \mid \vdash \Delta_2, B^\circ \in X \text{ and } A \equiv B^\perp \}$$

- define an object having good properties: $⫫$

  the set of cut-free provable sequents in $LK_\equiv$

- define an orthogonality operation on sets of sequents:

$$X^\perp = \{ \vdash \Delta, A^\circ \mid \vdash \Delta, A^\circ \star X \subseteq ⫫ \}$$

  - ★ usual properties of an orthogonality operation:

$$X \subseteq X^{\perp\perp} \qquad X \subseteq Y \Rightarrow Y^\perp \subseteq X^\perp \qquad X^{\perp\perp\perp} = X^\perp$$

Step 1: construct the set of reducibility candidates

- the domain of interpretation D: set of sequents

$$Ax^\circ \subseteq X \subseteq \perp\!\!\!\perp^\circ$$

which are behaviours: $X^{\perp\perp} = X$

- reducibility candidates analogy:

CR1 $X \subseteq \perp\!\!\!\perp$ (SN proofterms)
CR2 none (no reduction)
CR3 $Ax^\circ \subseteq X$ (neutral proofterms)

- core operation + orthogonality:

$$\begin{aligned} X.Y &= \{ \vdash \Delta_A, \Delta_B, (A \wedge B)^\circ \mid (\vdash \Delta_A, A^\circ) \in X \\ &\qquad \text{and } (\vdash \Delta_B, B^\circ) \in Y \} \\ X \wedge Y &= \{X.Y \cup Ax^\circ\}^{\perp\perp} \end{aligned}$$

Step 2: prove that it is a pre-Boolean algebra

D forms a pre-Boolean algebra:

- cheat on $\leq$: take the trivial pre-order
  - we can even drop it in the definition (see the paper)
- stability of *D* under $(.)^{\perp}$, $\wedge$
- stability of elements of *D* under $\equiv$

Step 3: prove adequacy

Super-consistency:

- give us an interpretation such that $A \equiv B$ implies $A^* = B^*$

Adequacy:

- takes a proof of $\vdash A_1, ..., A_n$
- assumes $\vdash \Delta_i, (A_i^\perp)^\circ \in A_i^{*\perp}$
- ensures $\vdash \Delta_1, ..., \Delta_n \in \perp\!\!\!\perp$

Features of the theorem:

- conversion rule: processed by the SC condition

Directly implies cut-elimination:

- because $Ax^\circ \subseteq A_i^{*\perp}$, we have $\vdash A, (A^\perp)^\circ \in A_i^{*\perp}$
- because of the definition of $\perp\!\!\!\perp$ (cut-free provable sequents)

As a conclusion...

- Deduction modulo defines a notion of analytic theories
- SC for pre-Heyting algebras implies normalization in $NJ_{\equiv}$
- SC for pre-Boolean algebras implies cut-elimination in $LK_{\equiv}$ using orthogonality
- SC for Heyting implies SC for Boole

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- Deduction modulo defines a notion of analytic theories
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some perspectives:

- does SC for Boole imply SC for Heyting ?
- what about double negative translations ?
- what about normalization in $LK_{\equiv}$?
- is SC complete w.r.t. normalization/cut-elimination ?