A Constructive Semantic Approach to Cut Elimination in Type Theories with Axioms

Olivier Hermant[1]* and James Lipton[2]

[1] *Univ. Complutense de Madrid*, Spain
ohermant@fdi.ucm.es
[2] *Wesleyan University*, USA
and visiting Researcher,
Univ. Politécnica de Madrid, Spain
jlipton@wesleyan.edu

Abstract. We give a fully constructive semantic proof of cut elimination for intuitionistic type theory with axioms. The problem here, as with the original Takeuti conjecture, is that the impredicativity of the formal system involved makes it impossible to define a semantics along conventional lines, in the absence, a priori, of cut, or to prove completeness by induction on subformula structure. In addition, unlike semantic proofs by Tait, Takahashi, and Andrews of variants of the Takeuti conjecture, our arguments are constructive.
Our techniques offer also an easier approach than Girard's strong normalization techniques to the problem of extending the cut-elimination result in the presence of axioms. We need only to relativize the Heyting algebras involved in a straightforward way.

1 Introduction

We give a new constructive semantic proof of cut elimination for an intuitionistic formulation of Church's Theory of Types (ICTT) with axioms. The argument extends and modifies techniques of Prawitz, Takahashi, Andrews and [4] which are non-constructive. A discussion of the constructive character of the proof, and the reasons why some older semantic proofs are not constructive can be found in Section 7. We also make use of a simple new technique to handle sets of axioms: relativization of infinite-context Heyting Algebras, as discussed below.

We recall that the central problem in extending the conventional syntactic proof of cut-elimination to certain *impredicative* higher-order logics is that one cannot induct on the natural subformula ordering that places instances $M[t/x]$ below quantified formulae such as $\exists x.M$, because it is not a well-ordering. This can be seen by taking M to be the variable x of type o and taking $t = \exists x.M \wedge A$ for any A, for example.

The problem of extending cut-elimination to higher-order logic (known as Takeuti's conjecture when it was still open) was solved by e.g. Takahashi[21],

* This work has been partially supported by the Spanish projects Merit-Forms-UCM (TIN2005-09207-C03-03) and Promesas-CAM (S-0505/TIC/0407).

Prawitz[18] and Andrews [1] by extending work by Tait [20] and following the blueprint given by Schütte in [19] where he observed that cut admissibility can be proved semantically by showing completeness of the cut-free fragment with respect to a weaker semantics he called semivaluations, and then showing every semivaluation gives rise to a total valuation extending it.

We generalize this approach by replacing Schütte's semivaluations by a *pair* of semantic mappings into a Heyting Algebra which give an upper and lower bound for the desired model, and show that such a pair can be defined syntactically (and constructively) using sets of contexts of cut-free proofs. The resulting model is easily relativized to extend to non-logical axioms by using a new parameter: an arbitrary set of axioms.

Cut-elimination for many impredicative formal systems (but not the ones considered here) has also been proved constructively using strong normalization techniques following Girard[8, 9]. We have chosen, rather, to take the alternative approach, namely that of the Takahashi-Schütte-Andrews tradition because it seems to lend itself more readily to the addition of axioms, a central concern of this paper. Also one of the main interests of the authors in this work is to apply these techniques to formal systems in which rewriting rules are combined with sequent calculus, such as Deduction Modulo, invented by Dowek, Hardin, Kirchner and Werner [5, 6]. Cut elimination for various fragments and variants of this system, studied elsewhere by Hermant and Dowek, does *not*, in general, satisfy strong normalization, and it is therefore not obviously amenable to Girard's techniques.

2 The Formal System: a Sketch

For definitions of types, terms and reduction in the intuitionistic formulation of Church's Theory of types, due originally to Miller et al. [13], we refer the reader to [2, 1, 4], and limit ourselves to recapitulating the rules of inference, in Fig. 1, where λ stands for $\beta\eta$ conversion, and where structural rules, such as contraction and weakening, are implicitly assumed. Types are omitted where clear from context, and we use Church's notation $(\beta\alpha)$ for the arrow type $\alpha \to \beta$ with association to the left. Fig. 1 does not include the cut rule:

$$\frac{\Gamma \vdash B \quad \Gamma, B \vdash A}{\Gamma \vdash A}\ \mathrm{Cut}$$

When we mean a proof within the rules of Fig. 1, we use the symbol $\vdash^*$, and the unadorned $\vdash$ when we allow the cut rule. $\Gamma \vdash A$ will also abbreviate "the sequent $\Gamma \vdash A$ has a proof". In the rest of the paper, we consider a fixed language S for ICTT, i.e. for each type, a set of constants.

3 Global Models

We will make use of the notion of applicative structures, a well-known semantic framework for the simply-typed lambda calculus [7, 17, 14].

$$\frac{}{\Gamma \vdash \top} \qquad \frac{}{\Gamma, U \vdash U} \qquad \frac{}{\Gamma, \bot \vdash \bot}$$

$$\frac{\Gamma, B, C \vdash A}{\Gamma, B \wedge C \vdash A}\ \wedge_L \qquad \frac{\Gamma \vdash B \quad \Gamma \vdash C}{\Gamma \vdash B \wedge C}\ \wedge_R$$

$$\frac{\Gamma, B \vdash A \quad \Gamma, C \vdash A}{\Gamma, B \vee C \vdash A}\ \vee_L \qquad \frac{\Gamma \vdash B_i}{\Gamma \vdash B_1 \vee B_2}\ \vee_R$$

$$\frac{\Gamma \vdash B \quad \Gamma, C \vdash A}{\Gamma, B \supset C \vdash A}\ \supset_L \qquad \frac{\Gamma, B \vdash C}{\Gamma \vdash B \supset C}\ \supset_R$$

$$\frac{\Gamma, P[t/x] \vdash A}{\Gamma, \forall x.P \vdash A}\ \forall_L \qquad \frac{\Gamma \vdash P}{\Gamma \vdash \forall x.P}\ \forall_R *$$

$$\frac{\Gamma, P \vdash A}{\Gamma, \exists x.P \vdash A}\ \exists_L * \qquad \frac{\Gamma \vdash P[t/x]}{\Gamma \vdash \exists x.P}\ \exists_R$$

$$\frac{\Gamma' \vdash A'}{\Gamma \vdash A}\ \lambda \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash B}\ \bot_R$$

Fig. 1. Higher-order Sequent Rules

Definition 1. *A typed applicative structure* $\langle \mathsf{D}, \mathsf{App}, \mathsf{Const}\rangle$ *consists of an indexed family* $\mathsf{D} = \{\mathsf{D}_\alpha\}$ *of sets* D_α *for each type* α*, an indexed family* App *of functions* $\mathsf{App}_{\alpha,\beta} : \mathsf{D}_{\beta\alpha} \times \mathsf{D}_\alpha \to \mathsf{D}_\beta$ *for each pair* (α, β) *of types, and an (indexed) interpretation function* $\mathsf{Const} = \{\mathsf{Const}_\alpha\}$ *taking constants of each type* α *to elements of* D_α*.*

We will abbreviate the mapping App to the infix operator $\cdot$ when types are clear from context.

So far we have only supplied a structure to interpret the underlying typed λ-calculus. Now we interpret the logic as well, by adjoining a Heyting algebra and some additional structure to handle the logical constants and predicates.

Definition 2. *A Heyting applicative structure, or HAS* $\langle \mathsf{D}, \mathsf{App}, \mathsf{Const}, \omega, \Omega\rangle$ *for* ICTT *is a typed applicative structure with an associated Heyting algebra* Ω *and function* ω *from* D_o *to* Ω *such that for each* f *in* $\mathsf{D}_{o\alpha}$*,* Ω *contains the parametrized meets and joins*

$$\bigwedge\{\omega(\mathsf{App}(f, d)) : d \in \mathsf{D}_\alpha\} \text{ and } \bigvee\{\omega(\mathsf{App}(f, d)) : d \in \mathsf{D}_\alpha\},$$

and the following conditions are satisfied:

$$\begin{aligned}
\omega(\mathsf{Const}(\top_o)) &= \top_\Omega \\
\omega(\mathsf{Const}(\bot_o)) &= \bot_\Omega \\
\omega(\mathsf{App}(\mathsf{App}(\mathsf{Const}(\wedge_{ooo}), d_1), d_2)) &= \omega(d_1) \wedge \omega(d_2) \\
\omega(\mathsf{App}(\mathsf{App}(\mathsf{Const}(\vee_{ooo}), d_1), d_2)) &= \omega(d_1) \vee \omega(d_2) \\
\omega(\mathsf{App}(\mathsf{App}(\mathsf{Const}(\supset_{ooo}), d_1), d_2)) &= \omega(d_1) \to \omega(d_2) \\
\omega(\mathsf{App}(\mathsf{Const}(\Sigma_{o(o\alpha)}), f)) &= \bigvee\{\omega(\mathsf{App}(f, d)) : d \in \mathsf{D}_\alpha\} \\
\omega(\mathsf{App}(\mathsf{Const}(\Pi_{o(o\alpha)}), f)) &= \bigwedge\{\omega(\mathsf{App}(f, d)) : d \in \mathsf{D}_\alpha\}
\end{aligned}$$

By supplying an object Ω of truth values we are able to distinguish between denotations of formulae (elements $d \in \mathsf{D}_o$) and their truth-values $\omega(d) \in \Omega$.[1]

An assignment φ is a function from the free variables of the language into D which respects types, and which allows us to give meaning to open terms.

Definition 3. *A global model for* ICTT *is a total assignment-indexed function* $\mathfrak{D} = \{\mathfrak{D}(\ )_\varphi : \varphi$ *an assignment*$\}$ *into an HAS (Heyting applicative structure)* $\langle \mathsf{D}, \mathsf{App}, \mathsf{Const}, \omega, \Omega \rangle$ *which takes (possibly open) terms of type* α *into* D_α *and satisfies the following environmental model conditions and* η*-conversion:*

$$\begin{array}{rll}
\mathfrak{D}(c)_\varphi = \mathsf{Const}(c) & & \textit{for constants } c \\
\mathfrak{D}(x)_\varphi = \varphi(x) & & \textit{for variables } x \\
\mathfrak{D}((MN))_\varphi = \mathsf{App}(\mathfrak{D}(M)_\varphi, \mathfrak{D}(N)_\varphi) & & \\
\mathfrak{D}(\lambda x_\alpha . M_\beta)_\varphi \quad \textit{is the unique member of } \mathsf{D}_{\beta\alpha} \textit{ s.t.} & & \\
\textit{for every } d \in \mathsf{D}_\alpha \ \mathsf{App}(\mathfrak{D}(\lambda x_\alpha . M_\beta)_\varphi, d) = \mathfrak{D}(M)_{\varphi[d/x]} & & \\
\mathfrak{D}(M)_\varphi = \mathfrak{D}(N)_\varphi & & M \ \eta\textit{-equivalent to } N
\end{array}$$

Given a model $\mathfrak{D}$ and an assignment φ, we say that φ satisfies B in $\mathfrak{D}$ if $\omega(\mathfrak{D}(B_o)_\varphi) = \top_\Omega$; this is abbreviated to $\mathfrak{D} \models_\varphi B_o$. We say B_o is valid in $\mathfrak{D}$ (equivalently, $\mathfrak{D} \models B_o$) if $\mathfrak{D} \models_\varphi B_o$ for every assignment φ. We abbreviate the truth-value $\omega(\mathfrak{D}(B_o)_\varphi)$ to $(B_o)^*_\varphi$. We also omit the subscript φ and parenthesis when our intentions are clear. We often use the word *model* just to refer to the mapping $(\_)^*$ from logical formulae to truth values in Ω.

3.1 Soundness of ICTT for Global Models

In the following we extend interpretations to sequents in a natural way.

Definition 4. *We define the meaning of a sequent in a model to be the truth-value in* Ω *given by:*

$$(\Gamma \vdash A)^* := (\bigwedge \Gamma \supset A)^*$$

where $\bigwedge \Gamma$ *signifies the conjunction of the elements of* Γ.

Note that $(\bigwedge \Gamma \supset A)^* = \top$ if and only if $\top \leq (\bigwedge \Gamma \supset A)^*$, which is equivalent to $(\bigwedge \Gamma)^* \leq (A)^*$. We will abbreviate $(\bigwedge \Gamma)^*$ to $(\Gamma)^*$, and express the validity of the indicated sequent by $(\Gamma)^* \leq (A)^*$ or, when referring to the environment, by $(\Gamma)^*_\varphi \leq (A)^*_\varphi$ henceforth.

Theorem 1 (Soundness). *If* $\Gamma \vdash A$ *is provable in* ICTT *then* $(\Gamma)^* \leq (A)^*$ *in every global model* $\mathfrak{E}$ *of* ICTT.

[1] This allows us to assign different truth values to $p_{oo}(A_o)$ and $p_{oo}(B_o)$ even when A and B are provably equivalent and hence have the same truth value. The equivalence of the higher order formulae $p_{oo}(A_o)$ and $p_{oo}(B_o)$ holds neither in ICTT as presented here nor in the λProlog programming language, based on a sub-system of ICTT.

A proof can be found in [4].

A straightforward proof of completeness of ICTT for global models can be given *under the assumption that cut is admissible for ICTT* along the lines of [22, 4], i.e. by choosing Ω to be the Lindenbaum algebra of equivalence classes of formulae and then interpreting each formula as its own equivalence class. Just to show Ω is partially ordered, we need cut.

Since we are not assuming cut holds in ICTT we must proceed differently. We will choose the complete Heyting algebra Ω_{cfk} generated by "relativized cut-free contexts", that is to say, contexts from which formulae can be proved without using cut. A partial valuation will be defined for this cHa, yielding an interpretation that establishes completeness and the admissibility of cut.

4 From Semivaluations to Valuations

In order to apply Schütte's plan [19], we need to extend the definition of a semivaluation in our intuitionistic (and higher-order) setting and lift the notion to Heyting Algebras. We also generalize Schütte's definition in one critical way: the partial information is given in terms of lower and an upper bounds for a model, which gives us an additional degree of freedom in how we approximate the truth value of a formula.

Definition 5. *Let Ω be a Heyting algebra. A global Ω* **semivaluation** $\mathcal{V} = \langle \mathsf{D}, \mathsf{App}, \mathsf{Const}, \pi, \nu, \Omega\rangle$ *consists of a typed applicative structure* $\langle \mathsf{D}, \mathsf{App}, \mathsf{Const}\rangle$ *together with a pair of maps $\pi : \mathsf{D}_o \longrightarrow \Omega$ and $\nu : \mathsf{D}_o \longrightarrow \Omega$, called the lower and upper constraints of $\mathcal{V}$, satisfying $\pi(d) \leq \nu(d)$ for any $d \in \mathsf{D}_o$, as well as the following:*

$$\begin{gathered}
\pi(\top_o) = \top_\Omega \qquad\qquad \pi(\bot_o) = \bot_\Omega \\
\pi(\mathsf{Const}(*) \cdot A \cdot B) \leq \pi(A) *_\Omega \pi(B) \qquad \textit{for } * \in \{\wedge, \vee, \supset\} \\
\pi(\mathsf{Const}(\Sigma_{o(o\alpha)}) \cdot f) \leq \bigvee\{\pi(f \cdot d) : d \in \mathsf{D}_\alpha\} \\
\pi(\mathsf{Const}(\Pi_{o(o\alpha)}) \cdot f_{(o\alpha)}) \leq \bigwedge\{\pi(f \cdot d) : d \in \mathsf{D}_\alpha\}
\end{gathered}$$

and

$$\begin{gathered}
\nu(\top_o) = \top_\Omega \qquad\qquad \nu(\bot_o) = \bot_\Omega \\
\nu(\mathsf{Const}(*) \cdot A \cdot B) \geq \nu(A) *_\Omega \nu(B) \qquad \textit{for } * \in \{\wedge, \vee, \supset\} \\
\nu(\mathsf{Const}(\Sigma_{o(o\alpha)}) \cdot f) \geq \bigvee\{\nu(f \cdot d) : d \in \mathsf{D}_\alpha\} \\
\nu(\mathsf{Const}(\Pi_{o(o\alpha)}) \cdot f_{(o\alpha)}) \geq \bigwedge\{\nu(f \cdot d) : d \in \mathsf{D}_\alpha\}
\end{gathered}$$

and the consistency *or* separation *conditions*

$$\pi(\mathsf{Const}(\supset) \cdot B \cdot C) \wedge \nu(B) \leq \pi(C) \tag{1}$$

$$\pi(B) \rightarrow_\Omega \nu(C) \leq \nu(\mathsf{Const}(\supset) \cdot B \cdot C). \tag{2}$$

Remark 1. The reader should note that some of these requirements are superfluous. For example, the separation conditions and the first condition imply the $\supset$ requirements for both π and ν. If we think of $[\pi(A), \nu(A)]$ as a – by definition nonempty – interval, one sees that it contains all the potential truth values of A, indeed the semantic "truth value candidates", instead of Girard's reducibility candidates. The circularity mentioned in the introduction will then be avoided by quantifying over all those candidates rather than subformulae.

The definition of environment, and global structure remain the same for semivaluations. As with Heyting applicative structures, in the presence of an environment φ, a semivaluation $\mathcal{V}$ induces an interpretation $\mathfrak{V}_\varphi$ from open terms A to the carriers D as follows:

$$\begin{array}{lll}
\mathfrak{V}(c)_\varphi & = \mathsf{Const}(c) & \text{for constants } c \\
\mathfrak{V}(x)_\varphi & = \varphi(x) & \text{for variables } x \\
\mathfrak{V}(M)_\varphi & = \mathfrak{V}(N)_\varphi & M \text{ eta-equivalent to } N \\
\mathfrak{V}((MN))_\varphi & = \mathsf{App}(\mathfrak{V}(M)_\varphi, \mathfrak{V}(N)_\varphi) & \\
\mathsf{App}(\mathfrak{V}(\lambda x_\alpha.M_\beta)_\varphi, d) & = \mathfrak{V}(M)_{\varphi[x:=d]} & \text{with } \mathfrak{V}(\lambda x_\alpha.M_\beta)_\varphi \text{ the unique such value.}
\end{array}$$

This assignment induces a *pair* of partial, or semi-truth-value assignments $[\![\_]\!]^\pi_\varphi$ and $[\![\_]\!]^\nu_\varphi$ to terms A_o of type o given by

$$\mathcal{V}[\![A]\!]^\pi_\varphi = \pi(\mathfrak{V}(A)_\varphi) \qquad\qquad \mathcal{V}[\![A]\!]^\nu_\varphi = \nu(\mathfrak{V}(A)_\varphi)$$

Theorem 2. *Given an Ω-semivaluation $\mathcal{V} = \langle \mathsf{D}, \cdot, \mathsf{Const}, \pi, \nu, \Omega\rangle$, there is a model $\mathfrak{D} = \langle \hat{\mathsf{D}}, \odot, \hat{\mathsf{C}}, \omega, \Omega\rangle$ extending $\mathcal{V}$ in the following sense: for all closed terms A_o*

$$\mathcal{V}[\![A]\!]^\pi \leq \omega(\mathfrak{D}(A)) \leq \mathcal{V}[\![A]\!]^\nu.$$

Furthermore, there is a surjective indexed map $\delta : \hat{\mathsf{D}} \longrightarrow \mathsf{D}$ such that for any $\hat{d} \in \hat{\mathsf{D}}_o$

$$\pi(\delta(\hat{d})) \leq \omega(\hat{d}) \leq \nu(\delta(\hat{d}))$$

Proof. We refer the reader to [4] for the proof.

5 Cut Elimination by Completeness

From Thm. 2, deriving a (cut-free) completeness theorem for ICTT requires a complete Heyting algebra Ω and an Ω-semivaluation. We first give the definition of Ω_{cfk}, the Heyting algebra of cut-free contexts, critically different from the one given in [4], where a tableaux-style construction is used, and extend the usual notion of context-based semantics [16, 15] to the notion of infinite contexts, taken themselves as a new free *parameter*.

5.1 The Cut-free Contexts Heyting Algebra

We first define what is a cut-free context, generalizing Okada [16, 15].

Definition 6 (outer value). *Assume given a set of formulae Ξ, possibly infinite, but containing only a finite number of variables. Let A be a closed formula. We let the* outer value *of A be:*

$$[\![A]\!] = \{\Gamma \mid \Xi, \Gamma \vdash^* A\}$$

The contexts Γ considered are always finite. The provability relation $\Xi, \Gamma \vdash^* A$ is with respect to some finite *subset of Ξ, Γ.*

So, an outer value $[\![A]\!]$ is the set of contexts proving A without cut (cut-free contexts). With this, we build Ω_{cfk}. When it is relevant, we stress the dependence on the considered set of axioms Ξ by $\Omega_{\mathsf{cfk}}(\Xi)$.

Definition 7 (Ω_{cfk}). *Let Ξ be a fixed set of formulae. Let $|\Omega|$ be the least set of sets of contexts generated by $[\![A]\!]$ for any formula A, closed under arbitrary (denumerable) intersection, and ordered by inclusion. Then define meets and joins on $|\Omega|$ as follows*

- $\bigwedge$ *= arbitrary intersection, just set-theoretic intersections.*
- $\bigvee$ *= arbitrary pseudo-union, that is to say*

$$\bigvee S = \bigcap \{c \in |\Omega| : c \geq S\}$$

where $c \geq S$ means $\forall s \in S\, c \geq s$

Remark 2. From the definition, it follows that:

- $\top_\Omega$ is the set of all contexts and as well $[\![\top_o]\!]$.
- $\bot_\Omega$ is the intersection of all $[\![A]\!]$ and as well $[\![\bot_o]\!]$. In particular, $\bot_\Omega \neq \emptyset$.
- the suprema can be slightly simplified: $a \vee_\Omega b = \bigcap \{[\![A]\!] \mid a \cup b \subseteq [\![A]\!]\}$, since any $c \in \Omega$ is of the form $\bigcap_{i \in \Lambda} [\![A_i]\!]$. As well, $\bigvee S = \bigcap \{[\![A]\!] \mid [\![A]\!] \geq S\}$.

Taking $a \to b = \bigvee \{x : x \wedge a \leq b\}$, the resulting structure $\Omega = \langle |\Omega|, \bigvee, \bigwedge, \to \rangle$ (also written Ω_{cfk}, when ambiguity may arise) is a complete Heyting algebra. We now check that the $\wedge \bigvee$ distributivity law [22] holds.

We first show one direction: for each member $a = \bigcap_i [\![A_i]\!]$ of Ω, we must have $a \cap \bigvee S \leq \bigvee a \cap S$, where $a \cap S$ means $\{a \cap s : s \in S\}$. Unfolding definitions, the inclusion to prove becomes:

$$a \cap \bigcap \{[\![B]\!] : [\![B]\!] \geq S\} \subseteq \bigcap \{[\![D]\!] : [\![D]\!] \geq a \cap S\} \tag{3}$$

Let Γ be a context member of the left hand side, i.e. such that $\Xi, \Gamma \vdash^* A_i$ for any A_i and $\Xi, \Gamma \vdash^* B$ for every B such that $[\![B]\!] \geq S$. Let D be a formula such that $[\![D]\!] \geq a \cap S$. We must show $\Xi, \Gamma \vdash^* D$ to prove that 3 holds.

Let Δ be a context such that $\Delta \in s$ for some $s \in S$. Since provability in Def. 6 deals with subcontexts, we directly have $\Xi, \Delta, \Gamma \vdash^* A_i$ and by a similar reasoning $\Delta, \Gamma \in s$. By definition of D, we get $\Xi, \Delta, \Gamma \vdash^* D$. Hence $\Delta \vdash^* \Gamma \supset D$, where $\Gamma \supset D$ is a shorthand for $B_1 \supset \cdots B_n \supset C$, and $\Delta \in [\![\Gamma \supset D]\!]$. Since this is valid for any s, we have shown $[\![\Gamma \supset D]\!] \geq S$.

But then, $\Xi, \Gamma \vdash^* \Gamma \supset D$ by assumption on Γ. By Kleene's Lem. 1 below and contraction on the formulae of Γ we have $\Xi, \Gamma \vdash^* D$, which shows that Γ is a member of the right-hand-side of 3, which proves the claim.

The other direction follows, by elementary lattice theory: for any $s \in S$ it is the case that $a \cap \bigvee S \geq a \cap s$. Now take the supremum of $a \cap s$ over all $s \in S$.

To complete the proof, we need Kleene's lemma, for the $\supset_R$ rule.

Lemma 1 (Kleene). *Let $C \equiv_\lambda A \supset B$ be formulae and Γ be a context. If $\Gamma \vdash C$ then $\Gamma, A \vdash B$, and if $\Gamma \vdash^* C$ then $\Gamma, A \vdash^* B$.*

Proof. Standard (see [10]) by induction on the structure of the proof.

5.2 A Semivaluation π and ν

Now, we need to exhibit a Ω-semivaluation to be able to apply Thm. 2. For this, we need the following definition:

Definition 8 (closure). *Let S be a set of contexts, we define its closure by:*

$$cl(S) = \bigcap\{[\![A]\!] \mid S \subseteq [\![A]\!]\}$$

It is the least element of Ω containing S. We also write, for a single context Γ, $cl(\Gamma)$ to mean $cl(\{\Gamma\})$.

Remark 3. $cl(A) \subseteq d$ is equivalent to $A \in d$ for any $d \in \Omega$. Indeed, $A \in cl(A)$ and $cl(A)$ is the l.u.b. of A. $cl(S)$ can also be seen as the *set of contexts admitting cut with all the elements of S* as shown in the following lemma.

Lemma 2. *Let A be a formula. The following formulations are equivalent:*

(i) $cl(A) = \bigcap\{[\![B]\!] \mid A \in [\![B]\!]\}$
(ii) $cl(A) = \{\Gamma \mid \Xi, \Gamma \vdash^* B$ *whenever* $\Xi, A \vdash^* B\}$. *Equivalently,* $\Gamma \in cl(A)$ *iff given any proof* $\Xi, A \vdash^* B$, *a proof of* $\Xi, \Gamma \vdash^* B$ *is derivable.*
(iii) $cl(A) = \{\Gamma \mid \Xi, \Gamma \vdash^* B$ *whenever* $\Xi, \Gamma, A \vdash^* B\}$. *Equivalently,* $\Gamma \in cl(A)$ *iff given any proof* $\Xi, \Gamma, A \vdash^* B$ *a proof of* $\Xi, \Gamma \vdash^* B$ *is derivable.*
(iv) $cl(A) = \{\Gamma \mid \Xi, \Delta, \Gamma \vdash^* B$ *whenever* $\Xi, \Delta, A \vdash^* B\}$. *Equivalently,* $\Gamma \in cl(A)$ *iff given any proof* $\Xi, \Delta, A \vdash^* B$ *a proof of* $\Xi, \Delta, \Gamma \vdash^* B$ *is derivable.*

Cases (ii) – (iv) *can be summarized as follows: Γ admits (Ξ-) cuts with A, hence the terminology "Γ is A-cuttable".*

Proof. (*ii*) unfolds the definition of $[\![B]\!]$ in (*i*). (*iii*) and (*iv*) reformulate (*ii*) with equivalent – thanks to Lem. 1, $\supset_R$ and contraction rules – notions of cuts.

We shall use any of the formulations given above, depending on our need. Now we are ready to give the semivaluation we work with.

Definition 9 (the cut-free context semivaluation). *Let the typed applicative structure $\langle \mathsf{D}, \mathsf{App}, \mathsf{Const}\rangle$ be the open term model: carriers D_α are open terms of type α in normal form, application $A \cdot B$ is $[AB]$, the normal form of AB, and we interpret constants as themselves. For any formula A, define:*

$$\pi(A) = cl(A) \quad \textit{and} \quad \nu(A) = [\![A]\!]$$

The definition just given of a pair of semantic mappings based on cut-free proofs and their contexts, and shown below to give rise to a semivaluation in the sense of Def. 5, is essential to the constructive character of our proof of cut-elimination, avoiding as it does the use of tableau style (Hintikka-set) construction of partial models, as in [1, 4], and the infinite tree arguments required.

Lemma 3. $\langle \mathsf{D}, \mathsf{App}, \mathsf{Const}, \pi, \nu, \Omega_{\mathsf{cfk}}\rangle$ *is a semivaluation in the sense of Def. 5.*

Proof. We check the conditions of Def. 5, with respect to the open term model. Each case follows the same pattern: it uses the corresponding rule of inference.

- $cl(A) \subseteq [\![A]\!]$. Immediate since $\{A\} \subseteq [\![A]\!]$ and from Rem. 3.
- $cl(\top_o) = \top_\Omega$. $\top_\Omega$ is the greatest element so we focus on the reverse inclusion. Consider a proof of $\Xi, \top_o \vdash^* A$. The only rule we can use on $\top_o$ besides structural ones and conversion is the axiom. We can replace it:

$$\dfrac{}{\vdash^* \top_o}\ \top_R$$

  Hence, $\Xi \vdash^* A$ and, by weakening, $\Xi, \Gamma \vdash^* A$ for any Γ, and $\top_\Omega \subseteq cl(\top_o)$.
- $cl(\bot_o) = \bot_\Omega$. $\bot_\Omega$ is the least element and, by other cases $cl(\bot) \subseteq [\![\bot]\!] = \bot_\Omega$.
- $cl(A \wedge B) \leq cl(A) \cap cl(B)$. This amounts to showing $A \wedge B \in cl(A) \cap cl(B)$. We prove that $A \wedge B$ is A-cuttable. Consider a proof of $\Xi, A \vdash^* C$. We construct the following proof:

$$\dfrac{\dfrac{\Xi, A \vdash^* C}{\Xi, A, B \vdash^* C}\ \text{weak}}{\Xi, A \wedge B \vdash^* C}\ \wedge_L$$

  Hence, $A \wedge B \in cl(A)$. On the same way, $A \wedge B \in cl(B)$.
- $cl(A \vee B) \subseteq cl(A) \vee_\Omega cl(B)$. It suffices to show $A \vee B \in cl(A) \vee_\Omega cl(B)$. Let C be such that $cl(A) \cup cl(B) \subseteq [\![C]\!]$. $A \in [\![C]\!]$, $B \in [\![C]\!]$, and the proof:

$$\dfrac{\Xi, A \vdash^* C \qquad \Xi, B \vdash^* C}{\Xi, A \vee B \vdash^* C}\ \vee_L$$

  shows that $A \vee B \in [\![C]\!]$. This holds for any such C, hence for their meet, and $A \vee B \in cl(A) \vee_\Omega cl(B)$.
- $cl(A \supset B) \subseteq cl(A) \to cl(B)$ is a consequence of $cl(A \supset B) \wedge [\![A]\!] \subseteq cl(B)$ (proved below) as mentioned in Rem. 1.
- $cl(\Sigma.f) \subseteq \bigvee\{cl(ft) \mid t \in \mathcal{T}_\alpha\}$ (where α is the suitable type). Equivalently, $\Sigma.f \in \bigvee\{cl((ft)) \mid t \in \mathcal{T}_\alpha\}$. Let t be a variable y of type α that is fresh for f and Ξ. We prove that $\Sigma.f$ is fy-cuttable. Assume we have a proof $\Xi, fy \vdash^* C$. The proof:

$$\frac{\Xi, fy \vdash^* C}{\Xi, \Sigma.f \vdash^* C} \exists_L$$

justifies the fy-cuttability. Hence $\Sigma.f \in cl(fy)$, and it is in the supremum.

- $cl(\Pi.f) \in \bigwedge\{cl(ft) \mid t \in \mathcal{T}_\alpha\}$. Let t be a term of type α. The proof:

$$\frac{\Xi, ft \vdash^* C}{\Xi, \Pi.f \vdash^* C} \forall_L$$

shows that $\Pi.f$ is ft-cuttable for any t.

- $[\![\top_o]\!] = \top_\Omega$ and $[\![\bot_o]\!] = \bot_\Omega$ hold both by definition, from Rem. 2.
- $[\![A \wedge B]\!] \supseteq [\![A]\!] \wedge_\Omega [\![B]\!]$. Let Γ such that $\Xi, \Gamma \vdash^* A$ and $\Xi, \Gamma \vdash^* B$. The claim is established by the proof:

$$\frac{\Xi, \Gamma \vdash^* A \qquad \Xi, \Gamma \vdash^* B}{\Xi, \Gamma \vdash^* A \wedge B} \wedge_R$$

- $[\![A \vee B]\!] \supseteq [\![A]\!] \vee_\Omega [\![B]\!]$. We show $[\![A \vee B]\!] \supseteq [\![A]\!]$. Let $\Gamma \in [\![A]\!]$. The proof:

$$\frac{\Xi, \Gamma \vdash^* A}{\Xi, \Gamma \vdash^* A \vee B} \vee_R$$

shows that $\Gamma \in [\![A \vee B]\!]$. Hence $[\![A \vee B]\!]$ is an upper bound for $[\![A]\!]$ and $[\![B]\!]$.

- $[\![A \supset B]\!] \supseteq [\![A]\!] \to_\Omega [\![B]\!]$ is a consequence of $cl(A) \to [\![B]\!] \subseteq [\![A \supset B]\!]$.
- $[\![\Sigma.f]\!] \supseteq \bigvee\{[\![ft]\!] \mid t \in \mathcal{T}_\alpha\}$. Let t be a term, and $\Gamma \in [\![ft]\!]$. The proof:

$$\frac{\Xi, \Gamma \vdash^* ft}{\Xi, \Gamma \vdash^* \Sigma.f} \exists_R$$

shows that $[\![\Sigma.f]\!]$ is an upper bound for any $[\![ft]\!]$, hence for their supremum.

- $[\![\Pi.f]\!] \supseteq \bigwedge\{[\![ft]\!] \mid t \in \mathcal{T}_\alpha\}$. Let $\Gamma \in \bigwedge\{[\![ft]\!] \mid t \in \mathcal{T}_\alpha\}$. Let y be a fresh variable with respect to Γ, Ξ and f. In particular, $\Gamma \in [\![fy]\!]$. The proof:

$$\frac{\Xi, \Gamma \vdash^* fy}{\Xi, \Gamma \vdash^* \Pi.f} \forall_R$$

shows that $\Gamma \in [\![\Pi.f]\!]$.

- $cl(B \supset C) \wedge_\Omega [\![B]\!] \subseteq cl(C)$. Let $\Gamma \in cl(B \supset C) \cap [\![B]\!]$. We must show the C-cuttability of Γ. Consider a proof of $\Xi, C \vdash^* D$. Since $\Gamma \vdash^* B$:

$$\frac{\Xi, \Gamma \vdash^* B \qquad \Xi, \Gamma, C \vdash^* D}{\Xi, \Gamma, B \supset C \vdash^* D} \supset_L$$

By $B \supset C$-cuttability of Γ we get $\Xi, \Gamma \vdash^* D$.

- $cl(B) \to_\Omega [\![C]\!] \subseteq [\![B \supset C]\!]$. Let $\Gamma \in cl(B) \to [\![C]\!]$ and show $\Xi, \Gamma \vdash^* B \supset C$. Since $\Gamma \in cl(B) \to [\![C]\!]$, we have $cl(\Gamma) \cap cl(B) \subseteq [\![C]\!]$. Furthermore, $\Gamma \in cl(\Gamma)$ and $B \in cl(B)$, therefore Γ, B belongs to both. So $\Gamma, B \in [\![C]\!]$, and we derive the desired proof:

$$\frac{\Xi, \Gamma, B \vdash^* C}{\Xi, \Gamma \vdash^* B \supset C} \supset_R$$

5.3 Completeness and Cut Elimination of ICTT

We now have all the results needed to establish completeness.

Theorem 3 (cut-free completeness of ICTT). *Let Γ be a context and A be a formula such that for any global model $\Gamma^* \leq A^*$. Then $\Gamma \vdash A$ has a cut-free proof.*

Proof. Calling ε the empty context, we apply Thm. 2 with the Heyting algebra $\Omega_{\mathsf{cfk}}(\varepsilon)$ given in Def. 7 and the semivaluation π, ν of Def. 9. We get, from Rem. 3, by Thm. 2 and by hypothesis that:

$$\Gamma \in cl(\Gamma) \subseteq \Gamma^* \subseteq A^* \subseteq [\![A]\!]$$

Hence, $\Gamma \vdash^* A$. An alternative proof involves $\Omega_{\mathsf{cfk}}(\Gamma)$: any context is trivially Γ-cuttable, so $\varepsilon \in cl(\Gamma) = \top$. With the same inclusions as above (but the first) we get that $\Gamma, \varepsilon \vdash^* A$. The interested reader may in fact prove Thm. 3 as many different ways than elements in $\mathfrak{P}(\Gamma)$, the powerset of Γ.

As an immediate corollary, we have:

Corollary 1 (constructive cut elimination for ICTT). *Let Γ be a context and A be a formula. If $\Gamma \vdash A$ in ICTT, then it has a proof without cut.*

Proof. By soundness and cut-free completeness, both of which were proved constructively.

6 Adding Non-logical Axioms

Now, we allow a more liberal notion of proof, with *non-logical axioms.*

Definition 10. *A non-logical axiom is a closed sequent $A \vdash B$. Assuming such and axiom $A \vdash B$, an axiomatic cut is the following implicit cut rule*

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma \vdash C}$$

A proof with non-logical axioms is a proof whose leaves are either a proper axiom rule, or a non-logical axiom and allowing the use of axiomatic cuts.

In the sequel, we fix a set (potentially infinite) of axioms, and consider proof system is ICTT with those non-logical axioms.

The constraint for $A \vdash B$ to be closed is not a theoretical limitation: it suffices to quantify over the free variables. In particular, we capture axiom schemes.

The two new rules overlap, since an axiomatic cut is simulated with a non-logical axiom and two (usual) cuts. Conversely, we can simulate the non logical axiom rules, even in a cut-free setting, so we often consider only axiomatic cuts:

$$\dfrac{\dfrac{}{\Gamma, A \vdash A} \qquad \dfrac{}{\Gamma, B \vdash B}}{\Gamma, A \vdash B}\ \text{axiomatic cut}$$

We show in this section that we still have, by the same means, cut elimination in ICTT with non-logical axioms, but that we can not, in the general setting, eliminate axiomatic cuts. First, we need another, unsurprising, notion of model:

Definition 11 (model for axioms). *A global model for* ICTT *(Def. 3) is a model of the non-logical axioms $A_i \vdash B_i, i \in \Lambda$ if and only if for any i, $A_i^* \leq B_i^*$.*

In the sequel, we will only be interested in such models.

Theorem 4 (Soundness of ICTT with non-logical axioms). *If $\Gamma \vdash A$ in* ICTT *with non-logical axioms, then $\Gamma^* \leq A^*$ in any global model of the non-logical axioms.*

Proof. We replace axiomatic cuts by axioms and cuts. Then the proof is done by the very same induction as the one of Thm. 1. The only additional case is a non-logical axiom $A \vdash B$, trivial from the assumption on the model.

Now we work towards a proof of a cut-free completeness theorem for ICTT with non-logical axioms. Cut-free means free of cuts, but not of axiomatic cuts, which we will not be able to remove.

6.1 Completeness and Cut Elimination in Presence of Axioms

Given the non logical axioms $A_i \vdash B_i$, let Ξ be the set of all the $A_i \supset B_i$. We show that the valuation in $\Omega_{\mathsf{cfk}}(\Xi)$ given by the Ω-semivaluation of Def. 9 is a model of the non-logical axioms. So in Lem. 4, provability refers to pure ICTT.

Lemma 4. *The valuation given by Theorem 2 with $cl(\_), [\![\_]\!]$ as an $\Omega_{\mathsf{cfk}}(\Xi)$-semivaluation is a model of the non-logical axioms.*

Proof. Let $A \vdash B$ be an axiom. $A^* \subseteq [\![A]\!]$ and $cl(B) \subseteq B^*$ by Thm. 2, so we show $[\![A]\!] \subseteq cl(B)$. This is implied by the fact that Γ is B-cuttable whenever $\Xi, \Gamma \vdash^* A$. Given a proof $\Xi, \Gamma, B \vdash^* C$, the following proof shows this claim:

$$\dfrac{\dfrac{\Xi, \Gamma, B \vdash^* C \qquad \Xi, \Gamma \vdash^* A}{\Xi, \Gamma, A \supset B \vdash^* C} \supset_L}{\Xi, \Gamma \vdash^* C} \text{contraction}$$

Before we prove the completeness theorem, we have to switch from proofs of $\Xi, \Gamma \vdash^* A$ in ICTT to proofs of $\Gamma \vdash^* A$ in ICTT with axiomatic cuts.

Lemma 5. *Assume we have a proof π of the sequent $\Gamma, A \supset B \vdash C$ in* ICTT, *possibly using axiomatic cuts. We can transform it into a proof of the sequent $\Gamma \vdash C$ in* ICTT *with additional cuts on the non logical axiom $A \vdash B$. If the initial proof is free of cuts, then so is the resulting proof (save axiomatic cuts).*

Proof. We can omit (by simulating) non logical axiom rules. We track the decomposition of $A \supset B$, and replace it by an axiomatic cut rule, that is the exact premises of the $\supset_L$ rule. We assume to have a proof of the sequent $\Gamma, D_1, \ldots, D_n \vdash C$, where $D_i \equiv_\lambda A \supset B$ and prove the result by induction over the structure of π. All cases are a trivial use of induction hypothesis, save:

– an axiom rule with D_1 the active formula. We build the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, A \vdash A}\text{ axiom} \quad \dfrac{}{\Gamma, B \vdash B}\text{ axiom}}{\Gamma, A \vdash B}\text{ axiomatic cut}}{\Gamma \vdash A \supset B}\supset_R}{\Gamma \vdash D_1}\lambda$$

– a $\supset$-l rule on $D_i = A' \supset B'$. We have the proof:

$$\dfrac{\dfrac{\pi_1}{\Gamma, D_2, \ldots, D_n \vdash A'} \quad \dfrac{\pi_2}{\Gamma, B', D_2, \ldots, D_n \vdash C}}{\Gamma, D_1, D_2, \ldots, D_n \vdash C}\supset_R$$

Applying induction hypothesis to get π_1' and π_2', we form the proof:

$$\dfrac{\dfrac{\dfrac{\pi_1'}{\Gamma \vdash A'}}{\Gamma \vdash A}\lambda \quad \dfrac{\dfrac{\pi_2'}{\Gamma, B' \vdash C}}{\Gamma, B \vdash C}\lambda}{\Gamma \vdash C}\text{ axiomatic cut}$$

Observe that we do not introduce any cut save axiomatic ones.

Theorem 5 (cut-free completeness of ICTT with non-logical axioms). *Let Γ be a context and A be a formula such that $\Gamma^* \leq A^*$ for any global model of the non logical axioms. Then $\Gamma \vdash A$ has a cut-free proof.*

Proof. Considering $\Omega_{\mathsf{cfk}}(\Xi)$ in Thm. 3, we get $\Xi, \Gamma \vdash^* A$ in ICTT. Applying Lem. 5 a finite number of times (provability is always with respect to a finite subset, from Def. 6), we get a cut-free proof of $\Gamma \vdash^* A$.

As an immediate corollary, we have:

Corollary 2 (constructive cut elimination for ICTT). *Let Γ be a context and A be a formula. If $\Gamma \vdash A$ has a proof in* ICTT *with non logical axioms, then it has a proof without cut.*

Proof. By soundness and cut-free completeness, both of which were proved constructively.

7 On the Constructivity of the Proof of Cut Admissibility

Our proof, unlike [21, 1] for the classical case or [4] for the intuitionistic case, makes no appeal to the excluded middle. The works cited (and our work as well) start directly, or indirectly from Schütte's observation [19] that cut admissibility can be proved semantically by showing completeness of the cut-free fragment with respect to semivaluations, and then showing every semivaluation gives rise to a total valuation extending it.

There are a number of pitfalls to avoid in finding a constructively valid proof based on this kind of argument, both in the way a semivaluation is produced and how one passes to a valuation.

Andrews shows [1] that any abstract consistency property gives rise to a semivaluation, but then builds one in a way that requires deciding whether or not a refutation exists of a given finite set of sentences. In particular, he needs to show (Thm. 3.5 in [1]) that *any finite set S satisfying an Abstract Consistency Property is consistent.* The proof actually establishes $\neg\neg$[Th. 3.5]. Furthermore, when showing that his cut-free proof system defines an Abstract Consistency Property (Sec. 4.10.2) he ends up proving the contrapositives of the defining properties of an ACP.

One can also exhibit a semivaluation by developing a tableau refutation of a formula (a Hintikka set) as is done in [4] but some care must be taken in the way the steps are formalized so as not to appeal to the fan theorem to produce an open path. No discussion of how this might be done appears in [4].

The proof given in this paper appeals to the strengthened version of Schütte's lemma in [4] which uses the more liberal definition of semivaluation *pairs*, (rather than semivaluations) which provide an upper and lower bound for the truth values of the valuation eventually produced by Takahashi's V-complex construction.

As we have shown, it is possible to give an instance of such a pair (namely $cl(\_)$ and $[\![\_]\!]$) without using tableaux and prove they satisfy the semivaluation axioms without appeal to the excluded middle.

Constructive Completeness. Producing a constructive proof of completeness is itself problematic, as pointed out by Gödel and discussed in [12, 23] if a sufficiently restrictive definition of validity is assumed, e.g. conventional Kripke models. However, there are a number of ways to liberalize the definition of validity to "save" constructive completeness [24, 3, 22, 11], in particular by allowing truth-values in a sufficiently broad class of structures. In our case these structures include complete Heyting Algebras *in which we cannot decide whether or not any given element is distinct from* $\top$ or even, for that matter, if the structure itself collapses to a one-element set. This appears to be a natural Heyting-valued counterpart to Veldman's exploding nodes [24].

In [22] completeness for an intuitionistic system *with* cut is shown constructively by mapping each formula to its own equivalence class in the Lindenbaum cHa. We cannot use this semantics here since cut is required to show that the target structure is partially ordered.

The semantics used in this paper can be seen as a cut-free variant of the Lindenbaum algebra, in which formulas are mapped to the sets of contexts that prove them without cut. Here too, one is not required to decide the provability of formulae in order to show model existence, in contrast with the $\top, \bot$-valued semantics of [1, 21].

References

1. Peter Andrews. Resolution in type theory. *Journal of Symbolic Logic*, 36(3), 1971.
2. A. Church. A formulation of the simple theory of types. *Journal of Symbolic Logic*, 5:56–68, 1940.

3. H. C.M̃. de Swart. Another intuitionistic completeness proof. *Journal of Symbolic Logic*, 41:644–662, 1976.
4. M. DeMarco and J. Lipton. Completeness and cut elimination in the intuitionistic theory of types. *Journal of Logic and Computation*, pages 821–854, November 2005.
5. Gilles Dowek, Thérèse Hardin, and Claude Kirchner. Theorem proving modulo. *Journal of Automated Reasoning*, 31:33–72, 2003.
6. Gilles Dowek and Benjamin Werner. Proof normalization modulo. *The Journal of Symbolic Logic*, 68(4):1289–1316, December 2003.
7. Harvey Friedman. Equality between functionals. In R. Parikh, editor, *Logic Colloquium*, volume 453 of *Lecture Notes in Mathematics*, pages 22–37. Springer, 1975.
8. J. Y. Girard. Une extension de l'interprétation de Gödel à l'analyse et son application à l'élimination de coupures dans l'analyse et la théorie des types. In J. E. Fenstad, editor, *Proceedings of the second Scandinavian proof theory symposium.* North–Holland, 1971.
9. J.Y. Girard, Y. Lafont, and P. Taylor. *Proofs and Types.* Cambridge University Press, 1998.
10. S. C. Kleene. Permutability of inferences in Gentzen's calculi LK and LJ. *Memoirs of the American Mathematical Society*, 10:1–26, 27–68, 1952.
11. Georg Kreisel. A remark on free choice sequences and the topological completeness proofs. *Journal of Symbolic Logic*, 23:369–388, 1958.
12. Georg Kreisel. On weak completeness of intuitionistic predicate logic. *Journal of Symbolic Logic*, 27:139–158, 1962.
13. Dale Miller, Gopalan Nadathur, Frank Pfenning, and Andre Scedrov. Uniform proofs as a foundation for logic programming. *Annals of Pure and Applied Logic*, 51(1-2):125–157, 1991.
14. John Mitchell. *Foundations for Programming Languages.* MIT Press, Cambridge, Massachusetts, 1996.
15. Mitsuhiro Okada. Phase semantic cut-elimination and normalization proofs of first- and higher-order linear logic. *Theoretical Computer Science*, 227:333–396, 1999.
16. Mitsuhiro Okada. A uniform semantic proof for cut-elimination and completeness of various first and higher order logics. *Theoretical Computer Science*, 281:471–498, 2002.
17. Gordon Plotkin. Lambda definability in the full type hierarchy. In J.P. Seldin and J. R. Hindley, editors, *To H.B. Curry: Essays in Combinatory Logic, Lambda Calculus and Formalism.* Academic Press, New York, 1980.
18. Dag Prawitz. Hauptsatz for higher order logic. *The Journal of Symbolic Logic*, 33(3):452–457, September 1968.
19. K. Schütte. Syntactical and semantical properties of simple type theory. *Journal of Symbolic Logic*, 25:305–326, 1960.
20. W. Tait. A non-constructive proof of Gentzen's Hauptsatz for second-order predicate logic. *Bulletin of the American Mathematical Society*, 72:980–983, 1966.
21. Moto-o Takahashi. A proof of cut-elimination in simple type theory. *J. Math. Soc. Japan*, 19(4), 1967.
22. A. S. Troelstra and D. van Dalen. *Constructivism in Mathematics: An Introduction*, volume 2. Elsevier Science Publishers, 1988.
23. Dirk van Dalen. *Lectures on Intuitionism*, pages 1 – 94. Number 337 in Lecture Notes in Mathematics. Springer Verlag, 1973.
24. W. Veldman. An intuitionistic completeness theorem for intuitionistic predicate logic. *Journal of Symbolic Logic*, 41:159–166, 1976.