A Model Based Cut Elimination Proof

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Abstract

We prove the cut elimination theorem for a wide range of rewrite systems in sequent calculus modulo. The sequent calculus modulo integrates in deduction steps some computations on propositions. We restrict our rewrite system to be confluent and compatible with a well-founded order. We give then an example of rewrite systems that fits theses conditions.

Introduction

The sequent calculus modulo, introduced in [4] is a deduction system based on the fact that some axioms can be successfully replaced by rewrite rules on terms an on propositions. This permits to have a faster proof-search and more readable proofs. But in general case, we loose the cut-elimination property that we had for the sequent calculus, the fact that the cut elimination property holds or not depends on the considered rewrite system (see below for counterexamples). On the other hand, we can express with the help of rewrite rules some powerful theories, such as Higher Order Logic [4] or Peano's arithmetic [5].

The paper [4] introduces a proof search system extending resolution for deduction modulo called Extended Narrowing And Resolution (ENAR). This method is proved complete for rewrite systems for which the cut elimination holds. Conversely, in [10] we have proved that the completness of ENAR implies the cut elimination property for sequent calculus modulo the same rewrite system.

Recent results of Stuber [11] prove the completness of ENAR using a semantic approach. Putting these two results together this leads to a new cut elimination theorem for the cases studied by Stuber [11]. In this paper, we give a direct proof of this cut elimination theorem avoiding the detour by the completness of ENAR.

In this paper, to prove the cut-elimination theorem, we prove the completness of the cut-free sequent calculus modulo, that is, Gödel's completness theorem. Conversely, we prove the soundness theorem that is for any provable sequent $\Gamma \vdash_{\mathcal{R}} \Delta$ (possibly with cuts), we have $\Gamma \models \Delta$ (where $\Gamma \models \Delta$

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means that for all models that validate Γ, at least one proposition of Δ is valid). These two results lead to the cut elimination theorem. Indeed, if we have $\Gamma \vdash_{\mathcal{R}} \Delta$, then, by soundness $\Gamma \models \Delta$ and then by th completness theorem $\Gamma \vdash_{\mathcal{R}}^{cf} \Delta$.

In order to prove the Gödel theorem for cut-free sequent calculus modulo, we will modify the proof based on Henkin witnesses. In a similar approach than in [11], we introduce an order and construct a model for a consistent, complete theory that admits Henkin witnesses. We have other modifications of the usual proof of the completness theorem due to the fact that we consider cut-free proofs. For example, when we have a consistent and complete theory, $\Gamma \vdash_{\mathcal{R}}^{cf} P$ does no more imply that $\Gamma, P \not\vdash_{\mathcal{R}}^{cf}$.

1 Deduction rules

1.1 Sequent calculus

Sequent calculus is a proof method for propositions that is equivalent to natural deduction, hence to proof *à la* Hilbert [2]. A sequent is written in the following way:

$$\Gamma \vdash \Delta$$

where Γ, Δ are multisets of propositions. The proof of a sequence is made with the help of deduction rules, that have the following shape:

$$\frac{A_1, ..., A_n}{B}$$

$A_1, ..., A_n, B$ are sequents, $A_1, ..., A_n$ are called premises and B the conclusion. Deduction rules for sequent calculus are presented in the figure 1.

In automatic reasoning, we start from the conclusion and we try to construct a proof tree for the conclusion. Sequent calculus is semi-decidable.

In the figure 1, we remark that the cut rule is the only rule where we have a proposition P in premises that doesn't appears in the conclusion. Thus, if we want to apply the cut rule, we are forced to test all the possible proposition P. Fortunately, a theorem, first proved by Gentzen [8], states that every sequent that has a proof has also a cut-free proof. This theorem implies the consistency of sequent calculus. Indeed, this is obvious that the sequent $\vdash$ has no cut-free proof. Hence by the cut elimination theorem it has no proof at all.

1.2 Sequent calculus modulo

Proofs contain many steps that seem self-evident, because they are based on computing, not on reasoning. Assume, for instance, that we are using Peano's axioms, and that we want to prove $\mathcal{PA} \vdash 2 + 2 = 4$. In this case, we would like to simplify the proposition $2 + 2 = 4$ into $4 = 4$ and then use the reflexivity axiom. The idea is to use a rewrite rule, rewriting on terms: $S(x) + y \to S(x + y)$, instead of the axiom $S(x) + y = S(x + y)$. The rewrite rules will not appear in the proof tree and thus proofs become more

$$\frac{}{\Gamma, P \vdash P, \Delta}\text{axiom}$$

$$\frac{\Gamma, P \vdash \Delta \quad \Gamma \vdash P, \Delta}{\Gamma \vdash \Delta}\text{cut}$$

$$\frac{\Gamma, P, P \vdash \Delta}{\Gamma, P \vdash \Delta}\text{contr-l}$$

$$\frac{\Gamma \vdash P, P, \Delta}{\Gamma \vdash P, \Delta}\text{contr-r}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, P \vdash \Delta}\text{weak-l}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash P, \Delta}\text{weak-r}$$

$$\frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \wedge\text{-l}$$

$$\frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \wedge\text{-r}$$

$$\frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta} \vee\text{-l}$$

$$\frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta} \vee\text{-r}$$

$$\frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \Rightarrow Q \vdash \Delta} \Rightarrow\text{-l}$$

$$\frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \Rightarrow Q, \Delta} \Rightarrow\text{-r}$$

$$\frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta}\neg\text{-l}$$

$$\frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta}\neg\text{-r}$$

$$\frac{}{\Gamma, \bot \vdash \Delta}\bot\text{-l}$$

$$\frac{\Gamma, \{t/x\}P \vdash \Delta}{\Gamma, \forall x P \vdash \Delta}\forall\text{-l}$$

$$\frac{\Gamma \vdash \{c/x\}P, \Delta}{\Gamma \vdash \forall x P, \Delta}\forall\text{-r, } c \text{ fresh constant}$$

$$\frac{\Gamma, \{c/x\}P \vdash \Delta}{\Gamma, \exists x P \vdash \Delta}\exists\text{-l, } c \text{ fresh constant}$$

$$\frac{\Gamma \vdash \{t/x\}P, \Delta}{\Gamma \vdash \exists x P, \Delta}\exists\text{-r}$$

Figure 1: Deduction rule of sequent calculus

compact and readable. Proof search is also improved, because we have to search a more compact proof. This system was formalized by G. Dowek, Th. Hardin, Cl. Kirchner [4]. Rules of the sequent calculus are preserved, but we have a more liberal condition between the premises and the conclusion, for instance, the $\wedge$-left rule is:

$$\frac{\Gamma, P, Q \vdash_{\mathcal{R}} \Delta}{\Gamma, R \vdash_{\mathcal{R}} \Delta} \wedge\text{-l, if } R =_{\mathcal{R}} (P \wedge Q)$$

The sequent calculus modulo is presented in the figure 2, with the help of the following definitions.

Definition 1 *A rewrite rule on a term is a pair of term $l \to r$ such that variables of r appears in l.*
A propositional rewrite rule is a pair of propositions $l \to r$ such that l is atomic and free variables of r appears in l.

Example of rewrite rule on a term:

$$x * 0 \to 0$$

Example of rewrite rule on atomic proposition:

$$x * y = 0 \to (x = 0) \vee (y = 0)$$

In this case, we notice that a non atomic proposition can rewrite on a non-atomic proposition.

A rewrite system written $\mathcal{R}$ is a set of propositional and term rewrite rules.

Definition 2 *Let $\mathcal{R}$ a rewrite system. The proposition P rewrites to P' in $\mathcal{R}$ iff:*
$P_{|\omega} = \sigma(l)$ and $P' = Q[\sigma(r)]_\omega$ for a rule $l \to r \in \mathcal{R}$, an occurence ω in P and a substitution σ. When we apply σ, we have to rename the bound variables in order to avoid capture.
We will write $P \to_{\mathcal{R}} P'$.

We write $\to^*_{\mathcal{R}}$ the transitive reflexive closure of $\to_{\mathcal{R}}$ and $=_{\mathcal{R}}$ its transitive, reflexive, symmetric closure.

In some case, we can add equationnal axioms to the rewrite rules, like the commutativity axiom:

$$x * y = y * x$$

We do not consider such cases here, although extending our results does not seem to be difficult. We just would have to reason on equivalence classes modulo equationnal axioms on terms and propositions.

We call the cut-free sequent calculus the calculus presented in 2 without the cut rule. We will consider this calculus from now. And we will now write a sequent:

$$\Gamma \vdash^{cf}_{\mathcal{R}} \Delta$$

$$\frac{}{\Gamma, P \vdash_{\mathcal{R}} Q, \Delta}\text{axiom, if } P =_{\mathcal{R}} Q$$

$$\frac{\Gamma, P \vdash_{\mathcal{R}} \Delta \quad \Gamma \vdash_{\mathcal{R}} Q, \Delta}{\Gamma \vdash_{\mathcal{R}} \Delta}\text{cut, if } P =_{\mathcal{R}} Q$$

$$\frac{\Gamma, Q_1, Q_2 \vdash_{\mathcal{R}} \Delta}{\Gamma, P \vdash_{\mathcal{R}} \Delta}\text{contr-l, if } P =_{\mathcal{R}} Q_1 =_{\mathcal{R}} Q_2$$

$$\frac{\Gamma \vdash_{\mathcal{R}} Q_1, Q_2, \Delta}{\Gamma \vdash_{\mathcal{R}} P, \Delta}\text{contr-r, if } P =_{\mathcal{R}} Q_1 =_{\mathcal{R}} Q_2$$

$$\frac{\Gamma \vdash_{\mathcal{R}} \Delta}{\Gamma, P \vdash_{\mathcal{R}} \Delta}\text{weak-l}$$

$$\frac{\Gamma \vdash_{\mathcal{R}} \Delta}{\Gamma \vdash_{\mathcal{R}} P, \Delta}\text{weak-r}$$

$$\frac{\Gamma, P, Q \vdash_{\mathcal{R}} \Delta}{\Gamma, R \vdash_{\mathcal{R}} \Delta} \wedge\text{-l, if } R =_{\mathcal{R}} (P \wedge Q)$$

$$\frac{\Gamma \vdash_{\mathcal{R}} P, \Delta \quad \Gamma \vdash_{\mathcal{R}} Q, \Delta}{\Gamma \vdash_{\mathcal{R}} R, \Delta} \wedge\text{-r, if } R =_{\mathcal{R}} (P \wedge Q)$$

$$\frac{\Gamma, P \vdash_{\mathcal{R}} \Delta \quad \Gamma, Q \vdash_{\mathcal{R}} \Delta}{\Gamma \vdash_{\mathcal{R}} \Delta} \vee\text{-l, if } R =_{\mathcal{R}} (P \vee Q)$$

$$\frac{\Gamma \vdash_{\mathcal{R}} P, Q, \Delta}{\Gamma \vdash_{\mathcal{R}} R, \Delta} \vee\text{-r, if } R =_{\mathcal{R}} (P \vee Q)$$

$$\frac{\Gamma \vdash_{\mathcal{R}} P, \Delta \quad \Gamma, Q \vdash_{\mathcal{R}} \Delta}{\Gamma, R \vdash_{\mathcal{R}} \Delta} \Rightarrow\text{-l, if } R =_{\mathcal{R}} (P \Rightarrow Q)$$

$$\frac{\Gamma, P \vdash_{\mathcal{R}} Q, \Delta}{\Gamma \vdash_{\mathcal{R}} R, \Delta} \Rightarrow\text{-r, if } R =_{\mathcal{R}} (P \Rightarrow Q)$$

$$\frac{\Gamma \vdash_{\mathcal{R}} P, \Delta}{\Gamma, R \vdash_{\mathcal{R}} \Delta}\neg\text{-l, if } R =_{\mathcal{R}} \neg P$$

$$\frac{\Gamma, P \vdash_{\mathcal{R}} \Delta}{\Gamma \vdash_{\mathcal{R}} R, \Delta}\neg\text{-r, if } R =_{\mathcal{R}} \neg P$$

$$\frac{}{\Gamma, P \vdash_{\mathcal{R}} \Delta}\bot\text{-l, if } P =_{\mathcal{R}} \bot$$

$$\frac{\Gamma, \{t/x\}P \vdash_{\mathcal{R}} \Delta}{\Gamma, Q \vdash_{\mathcal{R}} \Delta}\forall\text{-l, if } Q =_{\mathcal{R}} \forall x P$$

$$\frac{\Gamma \vdash_{\mathcal{R}} \{c/x\}P, \Delta}{\Gamma \vdash_{\mathcal{R}} Q, \Delta}\forall\text{-r, if } c \text{ fresh constant, and if } Q =_{\mathcal{R}} \forall x P$$

$$\frac{\Gamma, \{c/x\}P \vdash_{\mathcal{R}} \Delta}{\Gamma, Q \vdash_{\mathcal{R}} \Delta}\exists\text{-l if } c \text{ fresh constant and if } Q =_{\mathcal{R}} \exists x P$$

$$\frac{\Gamma \vdash_{\mathcal{R}} \{t/x\}P, \Delta}{\Gamma \vdash_{\mathcal{R}} Q, \Delta}\exists\text{-r, if } Q =_{\mathcal{R}} \exists x P$$

Figure 2: deduction rules for the sequent calculus modulo

Unfortunately, the cut elimination theorem does not extends to deduction modulo, for instance, it does not holds for the system defined by the following rule, which is a reformulation of Crabbé's counterexample for Naive Set Theory:

$$A \rightarrow B \wedge \neg A$$

With this rewrite rule, we can prove the proposition $\neg B$ under no assumption:

$$\vdash_{\mathcal{R}} \neg B$$

It is easy to see that this sequent doesn't have a cut-free proof.

Even for systems having good properties like confluence and termination, this is not always the case (see [6]). The cut elimination property then depends on the considered rewrite system.

Let's begin with some basic propositions.

Proposition 1 *Let $\Gamma, \Gamma', \Delta, \Delta'$ be multisets of propositions such that $\Gamma =_{\mathcal{R}} \Gamma'$ and $\Delta =_{\mathcal{R}} \Delta'$.*

$$\Gamma \vdash^{cf}_{\mathcal{R}} \Delta \quad \textit{iff} \quad \Gamma' \vdash^{cf}_{\mathcal{R}} \Delta'$$

Proof: by induction on the proof.

Let's consider the last rule. We can apply exactly the same on the premises obtained by induction hypotesis.
For example, if the last rule is $\exists$-left:

$$\cfrac{\cfrac{\pi}{\Gamma, Q[c/x] \vdash_{\mathcal{R}} \Delta}}{\Gamma, \exists x P \vdash_{\mathcal{R}} \Delta} \ \exists x Q =_{\mathcal{R}} P, c \text{ fresh constant}$$

then, we can find, by induction hypothesis, a proof:

$$\cfrac{\pi'}{\Gamma', Q[c/x] \vdash_{\mathcal{R}} \Delta'}$$

But c may be a non-fresh constant in the sequent $\Gamma', Q[c/x] \vdash_{\mathcal{R}} \Delta'$. This is not the good way to obtain this proof. Thus, we replace c in π by c', a fresh constant in π, Γ' and Δ'. By induction hypothesis, we have a proof of $\Gamma', Q[c'/x] \vdash_{\mathcal{R}} \Delta'$, with c' fresh in the sequent. We can now add the $\exists$-left rule:

$$\cfrac{\cfrac{\pi'}{\Gamma', Q[c'/x] \vdash_{\mathcal{R}} \Delta'}}{\Gamma', P' \vdash_{\mathcal{R}} \Delta'} \ \exists x Q =_{\mathcal{R}} P', c' \text{ fresh constant}$$

we have $\exists x Q =_{\mathcal{R}} P'$ by transitivity of the relation $=_{\mathcal{R}}$, because by hypothese $P =_{\mathcal{R}} P'$. □

We have the same result with the cut rule, but this is not of interest here.

Proposition 2 *Let Γ, Δ be two multisets of closed propositions. If $\Gamma \vdash^{cf}_{\mathcal{R}} \Delta$ has a proof, it has a closed proof.*

Proof: We can replace each free variable by a constant in the proof. □

2 Definitions

Definition 3 (Confluence) *Let $\mathcal{R}$ a rewrite system. He is said to be* confluent *if for all proposition P such that $P \rightarrow^*_{\mathcal{R}} P'$ and $P \rightarrow^*_{\mathcal{R}} P''$, there exists Q such that $P' \rightarrow^*_{\mathcal{R}} Q$ and $P'' \rightarrow^*_{\mathcal{R}} Q$.*

Definition 4 (Termination) *Let $\mathcal{R}$ a rewrite system. He is said to be* terminating *if and only if for any term there exists a finite rewrite sequence starting from this term and ending in a normal term.*

He is said to be strongly terminating *if and only if all the rewrite sequences are finite.*

If a rewrite system is terminating and confluent, then each proposition P has a unique normal form $\overline{P}$. In fact, by termination, there existst at least one normal form of P, say $P \rightarrow^* \overline{P}$. If $P \rightarrow^* P'$, then by confluence, there exists a proposition Q such that $P' \rightarrow^* Q$ and $\overline{P} \rightarrow^* Q$. By normality of $\overline{P}$, we have that $Q = \overline{P}$.

Definition 5 (Completness) *A theory Γ is said to be complete iff for any proposition P: $\Gamma, P \vdash^{cf}_{\mathcal{R}}$ or $\Gamma \vdash^{cf}_{\mathcal{R}} P$.*

This definition is slightly different from the classical definition of completness (we want that $\Gamma \vdash^{cf}_{\mathcal{R}} \neg P$ or $\Gamma \vdash^{cf}_{\mathcal{R}} P$). But it is equivalent under the hypothesis of confluence, thanks to lemma 6 below.

Definition 6 (Consistency) *A theory Γ is said to be consistent if and only if we can't prove the empty set of proposition.*
In our case Γ is consistent iff $\Gamma \nvdash^{cf}_{\mathcal{R}}$

Again, this definition is slightly different from the classical definitions of consistency:
Γ is inconsistent if and only if for any/at least one proposition P we have a proof of $\Gamma \vdash P$ and a proof of $\Gamma \vdash \neg P$.
This slight variation is important because we consider cut-free sequent calculus. It can be proved that our definition is a little bit stronger in the cut-free case.

Definition 7 *A theory Γ admits Henkin witnesses iff for any proposition Q with one free variable x, there exists a constant c of the language such that:*

$$\begin{array}{l} \Gamma, \exists x Q \nvdash^{cf}_{\mathcal{R}} \quad \textit{implies } Q[c/x] \in \Gamma \\ \textit{and} \\ \Gamma \nvdash^{cf}_{\mathcal{R}} \forall x Q \quad \textit{implies } \neg Q[c/x] \in \Gamma \end{array}$$

This definition is slightly different of the classical one too, for the reason as already explained.

Definition 8 (Model for a rewrite system) *Let $\mathcal{M}$ be a model. We say that it is a model of the rewrite rules if and only if for any propositions $P =_{\mathcal{R}} Q$ we have:*

$$\mathcal{M} \models P \textit{ iff } \mathcal{M} \models Q$$

In the rest of the paper, when we say model, we mean model of the rewrite system.

Definition 9 (Model for multisets) *Let Γ, Δ be multisets of propositions, and $\mathcal{M}$ a model. We say that $\mathcal{M}$ is a model of $\neg\Gamma, \Delta$ and we write:*

$$\mathcal{M} \models \neg\Gamma, \Delta$$

if there exists one proposition $P \in \Gamma$ such that $\mathcal{M} \models \neg P$ or if there exists one proposition $Q \in \Delta$ such that $\mathcal{M} \models Q$.

We write $\Gamma \models \Delta$ if for any model $\mathcal{M}$ we have $\mathcal{M} \models \neg\Gamma, \Delta$. This is equivalent to: in any model validating Γ, at least one proposition of Δ is valid.

The two notations are different formulation of the same thing. We will use the one or the other, depending of convenience of these notations.

3 Hypotheses

We consider the cut-free sequent calculus.

The language of propositions and terms is considered to be denumerable. In particular, we consider the set of constants to be denumerable.

We will admit the existence of an order on propositions that have the following properties:

- for any proposition $P = A\ \mathcal{C}\ B$ where $\mathcal{C}$ is a binary connector $(\wedge, \vee, \Rightarrow)$, $P \succ A$ and $P \succ B$,
- $\neg P \succ P$,
- $\exists x P \succ P[t/x]$ for any closed term t,
- $\forall x P \succ P[t/x]$ for any closed term t,
- let t and t' be two propositions or closed terms such that $t \rightarrow_{\mathcal{R}} t'$. $t \succ t'$ (compatibility of the rewrite system and the order),
- $\succ$ is a well-founded order.

We will suppose the rewrite system to be confluent, and compatible with such an order, thus strongly terminating. Hence the normal form of a proposition exists and is unique.

Let's start by the soundness theorem.

4 Soundness

Lemma 3 (Soundness) *Let $\mathcal{M}$ be any model of rewrite rules, in the sense of definition 9. If $\Gamma \vdash_{\mathcal{R}} \Delta$ (with possible cuts), then $\mathcal{M} \models \neg\Gamma, \Delta$.*

Proof: We begin by completing the language $\mathcal{L}$ in adding a set of new constants, $\mathcal{L}_{\mathcal{M}}$, composed of constants of the type c_a, where a in an object of the model $\mathcal{M}$. We extend the model $\mathcal{M}$ interpreting a constant c_a by its associated object $a \in \mathcal{M}$. The truth value of propositions in $\mathcal{L}$ and the interpretation of terms in $\mathcal{L}$ remain unchanged.

Now, we make an induction on the proof (that is considered in the language $\mathcal{L} \cup \mathcal{L}_{\mathcal{M}}$). Let's consider some key cases. If the last rule is:

- $\vee$-left:

$$\frac{\dfrac{\pi}{\Gamma, A \vdash_{\mathcal{R}} \Delta} \quad \dfrac{\pi'}{\Gamma, B \vdash_{\mathcal{R}} \Delta}}{\Gamma, A \vee B \vdash_{\mathcal{R}} \Delta}$$

  By induction hypothesis $\mathcal{M} \models \neg\Gamma, \neg A, \Delta$, and $\mathcal{M} \models \neg\Gamma, \neg B, \Delta$. We have either $\mathcal{M} \models \neg\Gamma, \Delta$, either $\mathcal{M} \not\models A$ and $\mathcal{M} \not\models B$, hence $\mathcal{M} \not\models A \vee B$. In both case: $\mathcal{M} \models \neg\Gamma, \neg A \vee B, \Delta$.

- $\wedge$-left:

$$\frac{\dfrac{\pi}{\Gamma, A, B \vdash_{\mathcal{R}} \Delta}}{\Gamma, A \wedge B \vdash_{\mathcal{R}} \Delta}$$

  By induction hypothesis, $\mathcal{M} \models \neg\Gamma, \neg A, \neg B, \Delta$. Either $\mathcal{M} \models \neg\Gamma, \Delta$, either $\mathcal{M} \models \neg A, \neg B$ and thus $\mathcal{M} \not\models A$ or $\mathcal{M} \not\models B$, that implies $\mathcal{M} \not\models A \wedge B$. In any case: $\mathcal{M} \models \neg\Gamma, \neg A \wedge B, \Delta$.

- $\forall$-left:

$$\frac{\dfrac{\pi}{\Gamma, P[t/x] \vdash_{\mathcal{R}} \Delta}}{\Gamma, \forall x P \vdash_{\mathcal{R}} \Delta}$$

  By induction hypothesis, $\mathcal{M} \models \neg\Gamma, \neg P[t/x], \Delta$. We have two possibilities. The first, if $\mathcal{M} \models \neg\Gamma, \Delta$ we have directly $\mathcal{M} \models \neg\Gamma, \neg\forall x P, \Delta$.The second possibility is $\mathcal{M} \models \neg P[t/x]$, that is $\mathcal{M} \not\models P[t/x]$, that is $\mathcal{M} \not\models \forall x P$, and the proof is finished.

- $\exists$-left:

$$\frac{\dfrac{\pi}{\Gamma, P[c/x] \vdash_{\mathcal{R}} \Delta}}{\Gamma, \exists x P \vdash_{\mathcal{R}} \Delta}$$

  c is a fresh constant. Replacing some constants in the proof π by other fresh constants, we can replace the constant c by any constant c_a:

$$\frac{\pi}{\Gamma, P[c_a/x] \vdash_{\mathcal{R}} \Delta}$$

  By induction hypothesis, we have, for any constant c_a: $\mathcal{M} \models \neg\Gamma, \neg P[c_a/x], \Delta$. Either we have $\mathcal{M} \models \neg\Gamma, \Delta$ and the proof is finished, either we have, for any constant closed term $\mathcal{M} \not\models P[c_a/x]$. This is exactly the definition of $\mathcal{M} \not\models \exists x P$. Hence, we have $\mathcal{M} \models \neg\Gamma, \neg\exists x P, \Delta$.

- Cut: By induction hypothesis, we have:

$$\mathcal{M} \models \neg\Gamma, \neg C, \Delta$$
$$\mathcal{M} \models \neg\Gamma, D, \Delta$$

  Remembering that $C =_{\mathcal{R}} D$, we can't have at the same time $\mathcal{M} \models \neg C$ and $\mathcal{M} \models D$. Hence, there exists a proposition P in $\neg\Gamma, \Delta$ such that $\mathcal{M} \models P$

- All the other cases are similar.

□

Corollary 4 *Let $\mathcal{M}$ be a model of rewrite rules, in the sense of definition 9. If $\Gamma \vdash_{\mathcal{R}}^{cf} \Delta$, then $\mathcal{M} \models \neg\Gamma, \Delta$.*

Proof: Exactly the same as lemma 3. We just have to consider one case less: if the last rule is a cut rule. □

In the remainder of the paper, we shall prove the converse of this corollary 4, namely completness of the cut-free sequent calculus modulo. We start by a few basic lemmas.

5 Basic results

Lemma 5 *Let P, Q be two non-atomic propositions such that $P =_{\mathcal{R}} Q$. Then P and Q have the same main connector.*

Proof: Confluence of the rewrite system claims that there exists a proposition R such that $P \rightarrow^* R$ and $Q \rightarrow^* R$. As propositional rewrite rules are only on atoms, P and R have the same main connector. We have the same result for Q and R. □

The following lemma is an extension of a Kleene result [9].

Lemma 6 (Kleene) *Let Γ, Δ be multisets of propositions, and $A_1, ..., A_n, P_1, ..., P_n$ propositions such that $A_1 =_{\mathcal{R}} \neg P_1$, ..., $A_n =_{\mathcal{R}} \neg P_n$. If*

$$\Gamma, A_1, ..., A_n \vdash_{\mathcal{R}}^{cf} \Delta$$

then we can construct a proof:

$$\Gamma \vdash_{\mathcal{R}}^{cf} P_1, ..., P_n, \Delta$$

Proof: By induction over the proof structure.

If the last rule is a rule on Γ, Δ, we apply the induction hypothesis on the premises, to obtain some proofs on which we apply the same rule.

- If the last rule is a weak-left rule on A_1:

$$\frac{\Gamma, A_2, ..., A_n \vdash_{\mathcal{R}}^{cf} \Delta}{\Gamma, A_1, A_2, ..., A_n \vdash_{\mathcal{R}}^{cf} \Delta}$$

  then we apply the induction hypothesis on premise, and on the obtained proof, we apply the weak-right rule:

$$\frac{\Gamma \vdash_{\mathcal{R}}^{cf} P_2, ..., P_n, \Delta}{\Gamma \vdash_{\mathcal{R}}^{cf} P_1, P_2, ..., P_n, \Delta}$$

- If the last rule is a contraction rule on A_1:

$$\frac{\Gamma, A_1', A_1, A_2, ..., A_n \vdash_{\mathcal{R}}^{cf} \Delta}{\Gamma, A_1, ..., A_n \vdash_{\mathcal{R}}^{cf} \Delta}$$

  $A_1' =_{\mathcal{R}} A_1 =_{\mathcal{R}} \neg P_1$. We apply induction hypothesis on premise, and on the obtained proof, we apply the contraction-right rule:

$$\frac{\Gamma \vdash_{\mathcal{R}}^{cf} P_1, P_1, P_2, ..., P_n, \Delta}{\Gamma \vdash_{\mathcal{R}}^{cf} P_1, ..., P_n, \Delta}$$

- If it is a $\neg$-left rule, we apply the induction hypothesis on premise, and we directly found the wanted proof.

- If it is an axiom rule on the sequent $\Gamma, A_1, ..., A_n \vdash_{\mathcal{R}}^{cf} \Delta$ we replace it by the following proof:

$$\frac{\overline{\Gamma, P_1 \vdash_{\mathcal{R}}^{cf} P_1, ..., P_n, \Delta'}\ \text{axiom}}{\Gamma \vdash_{\mathcal{R}}^{cf} P_1, ..., P_n, B, \Delta'}\ \neg\text{-right}$$

  with $\Delta = B, \Delta'$ and $B =_{\mathcal{R}} A_1 =_{\mathcal{R}} \neg P_1$.

- we can't have other rule on A_p, because if we have $B =_{\mathcal{R}} A_1 =_{\mathcal{R}} \neg P_1$, B is atomic of its main connector is $\neg$ by lemma 5

□

We now prove the key lemma, that will be used for the model construction. It says that the cut rule is redundant for normal atomic atoms. It implies that we can not have at the same time $\Gamma, A \vdash_{\mathcal{R}}^{cf}$ and $\Gamma \vdash_{\mathcal{R}}^{cf} A$ for a normal atom and a consistent theory Γ, despite the fact that we don't have the cut rule.

Lemma 7 (Normal Atomic Cut Elimination) *Let Γ, Δ be multisets of propositions, and A a normal atom. If we have:*

$$\Gamma, A_1, ..., A_n \vdash_{\mathcal{R}}^{cf} \Delta$$
$$\Gamma \vdash_{\mathcal{R}}^{cf} A, \Delta$$

with $A_1 =_{\mathcal{R}} A_2 =_{\mathcal{R}} ... =_{\mathcal{R}} A_n =_{\mathcal{R}} A$, we can construct a proof of:

$$\Gamma \vdash_{\mathcal{R}} \Delta$$

Proof: Propositions $A_1, ..., A_n$ are atomic. By confluence, there exists P such that $A_1 \rightarrow^*_{\mathcal{R}} P$, ..., $A_n \rightarrow^*_{\mathcal{R}} P$, $A \rightarrow^*_{\mathcal{R}} P$. A is a normal atom, hence $P = A$.

We add to each sequent of the proof $\Gamma, A_1, ..., A_n \vdash_{\mathcal{R}} \Delta$ multisets of propositions Γ_* and Δ_* (equals respectively to Γ, Δ), in order to get a proof of $\Gamma_*, \Gamma, A_1, ..., A_n \vdash^{cf}_{\mathcal{R}} \Delta, \Delta_*$. propositions of Γ_* and Γ and of Δ_* and Δ will be distinguished by a subscripted star.

At the leaves of the proof, we now have axiom rules:

$$\frac{}{\Gamma_*, \Gamma', A \vdash^{cf}_{\mathcal{R}} B, \Delta', \Delta_*} \text{ axiom}$$

Γ', Δ' may be empty, and $A =_{\mathcal{R}} B$.

Thus, we have by hypothesis, if we remember that $\Gamma_* = \Gamma$ and $\Delta_* = \Delta$:

$$\Gamma_*, \Gamma', A_1, ..., A_n \quad \vdash^{cf}_{\mathcal{R}} \quad \Delta', \Delta_* \tag{1}$$

$$\Gamma_* \quad \vdash^{cf}_{\mathcal{R}} \quad A, \Delta_* \tag{2}$$

with $\Gamma' = \Gamma$ and $\Delta' = \Delta$.

We construct now by induction on proof 1 a proof of:

$$\Gamma_*, \Gamma' \vdash^{cf}_{\mathcal{R}} \Delta', \Delta_*$$

We consider the last rule:

- If it is a rule on propositions of Γ', Δ', by induction hypothesis, we have a proof of $\Gamma_*, \Gamma'' \vdash^{cf}_{\mathcal{R}} \Delta'', \Delta_*$ to which we can apply the same rule.

- If it is a weak-left or a contract-left rule on A_p, we apply the induction hypothesis.

- If it is an axiom rule on A_p, then it is of the following shape:

  $$\Gamma_*, \Gamma', A_1, ..., A_p, ..., A_n \vdash^{cf}_{\mathcal{R}} \Delta', \Delta_*$$

  Some proposition P congruent to A_p (thus to A) must appear in Δ' (see definition of the axiom rule, figure 2). By proposition 1, from the proof 2 we construct a proof of $\Gamma_* \vdash^{cf}_{\mathcal{R}} P, \Delta_*$. Adding weak rules we get a proof of $\Gamma_*, \Gamma' \vdash^{cf}_{\mathcal{R}} \Delta', \Delta_*$.

We notice that we can't have other rules on A_p, because its congruence class is formed of atomic proposition, as proved at the beginning of this proof. Especially, we don't have any connector rule.

We contructed a proof of $\Gamma_*, \Gamma \vdash^{cf}_{\mathcal{R}} \Delta, \Delta_*$. As $\Gamma_* = \Gamma$ et $\Delta_* = \Delta$, we can now use many contraction rule, and obtain a proof of what we wanted:

$$\Gamma \vdash^{cf}_{\mathcal{R}} \Delta$$

□

6 Construction of a theory and a model

Given a consistent theory $\mathcal{T}$, we are going to construct a complete, consistent theory Γ that admits Henkin witnesses, according to the definition 7. We then will construct a Herbrand model for Γ.

6.1 Construction of the theory

We consider a consistent theory $\mathcal{T}$ expressed in a denumerable language $\mathcal{L}$. We add to this language a denumerable set of new constants $\mathcal{C}$.

Definition 10 (Theory construction) *We let $\Gamma_0 = \mathcal{T}$, and we enumerate all the proposition of the language:*

$$P_0, ..., P_n, ...$$

We define Γ_{n+1} by induction:

- *If $P_n = \exists x Q$ and if $\Gamma_n, \exists x Q \not\vdash^{cf}_{\mathcal{R}}$ then let $c \in \mathcal{C}$ a constant that doesn't appear in Γ_n. This is possible, because a finite number (at most n) of these constants appear in Γ_n. We let $\Gamma_{n+1} = \Gamma_n \cup \{Q[c/x], P_n\}$.*
- *If $\Gamma_n, P_n \not\vdash^{cf}_{\mathcal{R}}$ then $\Gamma_{n+1} = \Gamma_n \cup \{P_n\}$*
- *If $P_n = \forall x Q$, if $\Gamma_n, \forall x Q \vdash^{cf}_{\mathcal{R}}$ and if $\Gamma_n \not\vdash^{cf}_{\mathcal{R}} \forall x Q$ then we let $\Gamma_{n+1} = \Gamma_n \cup \{\neg Q[c/x]\}$, c being a fresh constant of $\mathcal{C}$.*
- *Else, we let $\Gamma_{n+1} = \Gamma_n$.*

At last, we take $\Gamma = \bigcup_{n=0}^{\infty} \Gamma_n$.

Let's prove some properties of Γ:

- Γ is complete: let P be a proposition, then we know that there exists a number n such that $P_n = P$ in our enumeration.

  Either $\Gamma_n, P_n \vdash^{cf}_{\mathcal{R}}$ and this is immediate that $\Gamma, P \vdash^{cf}_{\mathcal{R}}$, either $\Gamma_n, P_n \not\vdash^{cf}_{\mathcal{R}}$ and in this case, $P_n \in \Gamma_{n+1} \subset \Gamma$, hence $\Gamma \vdash^{cf}_{\mathcal{R}} P$ with the axiom rule. $\Box$

- Γ is consistent: suppose $\Gamma \vdash_{\mathcal{R}}$, it means that there exists a finite subset $\Gamma' \subset \Gamma$ such that $\Gamma' \vdash_{\mathcal{R}}$.

  Let N be the greatest index of a proposition in Γ'. Then we have $\Gamma' \subset \Gamma_N$, hence $\Gamma_N \vdash_{\mathcal{R}}$.

  We take the least N such that $\Gamma_N \vdash_{\mathcal{R}}$. $N \neq 0$, because $\mathcal{T}$ is consistent. $\Gamma_N \neq \Gamma_{N-1}$, because N is the least index. Looking back at the definition 10, we see that we are in one of the following case:

- $P_{N-1} = \forall x Q$, $\Gamma_{N-1}, P_{N-1} \vdash^{cf}_{\mathcal{R}}$ and $\Gamma_{N-1} \not\vdash^{cf}_{\mathcal{R}} P_{N-1}$.
  We have $\Gamma_{N-1}, \neg Q[c/x] \vdash^{cf}_{\mathcal{R}}$ by construction of Γ_N and by hypothese, and we conclude that we have $\Gamma_{N-1} \vdash^{cf}_{\mathcal{R}} Q[c/x]$ by lemma 6. The constant c is fresh with respect to Γ_{N-1} (by construction 10). We can thus apply the $\forall$-right rule and get a proof of $\Gamma_{N-1} \vdash^{cf}_{\mathcal{R}} \forall x Q$.
- $P_{N-1} = \exists x Q$ and $\Gamma_{N-1}, P_{N-1} \not\vdash^{cf}_{\mathcal{R}}$. We have $\Gamma_{N-1}, P_{N-1}, Q[c/x] \vdash^{cf}_{\mathcal{R}}$ by definition of Γ_N and by hypothese. c being a fresh constant by construction, we can apply the rule $\exists$-left, then the contraction-left rule, and we get $\Gamma_{N-1}, P_{N-1} \vdash^{cf}_{\mathcal{R}}$.
- Else, we have $\Gamma_{N-1}, P_{N-1} \not\vdash^{cf}_{\mathcal{R}}$, and, by hypothese on Γ_N, $\Gamma_{N-1}, P_{N-1} \vdash^{cf}_{\mathcal{R}}$.

In all the three cases we get that $\Gamma_{N-1}, P_{N-1} \not\vdash^{cf}_{\mathcal{R}}$ and $\Gamma_{N-1}, P_{N-1} \vdash^{cf}_{\mathcal{R}}$, that is a contradiction. □

- Γ admits Henkin witnesses. Let $P = \exists x Q$ be a proposition such that $\Gamma, P \not\vdash^{cf}_{\mathcal{R}}$. Let N be the number such that $P_N = P$. Then $\Gamma_N, P \not\vdash_{\mathcal{R}}$, because $\Gamma_N \subset \Gamma$.

  Hence, there exists a constant $c \in \mathcal{C}$ such that $Q[c/x] \in \Gamma_{N+1} \subset \Gamma$.

  Let now $P = \forall x Q$ be a proposition such that $\Gamma \not\vdash^{cf}_{\mathcal{R}} P$. Let N be the index of P in the enumeration. Then we have $\Gamma_{N+1} \not\vdash^{cf}_{\mathcal{R}} P$. In particular Γ_{N+1} doesn't contain P. Hence we can't have $\Gamma_N, P_N \not\vdash_{\mathcal{R}}$ (see definition 10). In other word:

$$\Gamma_N, P \vdash_{\mathcal{R}}$$
$$\Gamma_N \not\vdash_{\mathcal{R}} P$$

  the second result is obtain because $\Gamma_N \subset \Gamma$. Hence, by construction there exists a constant c such that $\neg Q[c/x] \in \Gamma_{N+1} \subset \Gamma$.

□

6.2 Construction of the model

We now construct the model, like in [11].
We only consider the closed propositions of the language. For the domain of the model we take all closed normal terms. We are going to define the truth value for each atom, begining with normal atoms.

We interpret a term by its normal form. This is consistent with the definitions below.

For each normal atom, we let:

$$\begin{cases} \text{if } \Gamma, A \vdash^{cf}_{\mathcal{R}}, |A|_{\mathcal{M}} = 0 \text{ (False)} \\ \text{if } \Gamma \vdash^{cf}_{\mathcal{R}} A, |A|_{\mathcal{M}} = 1 \text{ (True)} \end{cases}$$

This definition is valid, because lemma 7 asserts that we can't have the two cases at the same time for a consistent theory.

For each normal proposition, we now construct its tree.

The tree for the proposition $A \vee B$ is a tree whose root is $\vee$ and whose sons are the two trees of the normal forms of A and B.
The tree for the proposition $A \wedge B$ is a tree whose root is $\wedge$ and whose sons are the two trees of the normal forms of A and B.
The tree for the proposition $A \Rightarrow B$ is a tree whose root is $\Rightarrow$ and whose left son is the tree of A and right son is the tree of B
The tree for the proposition $\neg A$ is a tree whose root is $\neg$ and whose son is the tree of the normal forms of A.
The tree of the normal proposition $\exists x A$ is a tree whose root is labeled $\exists$ and whose sons are the trees of the normal forms of $A[t/x]$, for each closed term t.
The tree of the normal proposition $\forall x A$ is a tree whose root is labeled $\forall$ and whose sons are the trees of the normal forms of $A[t/x]$, for each closed term t.

The tree is finite, because each time we go down in the tree, we decrease the well-founded order $\succ$.

To a tree whose root is $\vee$, we assign the True truth value if and only if at least one of its sons has the True thruth value, else we assign the false truth value.
To a tree whose root is $\wedge$, we assign the True truth value if and only if both of its sons has the True thruth value, else we assign the False truth value.
To a tree whose root is $\Rightarrow$, we assign the True truth value if and only if the left son has the False truth value or if the right son has the True truth value, else we assign the false truth value.
To a tree whose root is $\neg$, we assign the True truth value if and only if its son has the False thruth value, else we assign the False truth value.
We assign the True truth value to a tree whose root is $\exists$ if and only if at least one of its sons has the True truth value.
We assign the True truth value to a tree whose root is $\forall$ if and only if all of its sons has the True truth value.

The truth value of a non-normal proposition is the truth value of its normal form.

We can easily verify that the model $\mathcal{M}$ is a model of the rewrite rules, and is such that for each normal atom:

$$\Gamma, A \not\vdash_{\mathcal{R}}^{cf} \text{ iff } \mathcal{M} \models A$$

We prove now that this result extends to all propositions. More generally, if we have a model of the rewrite rules verifying this property, this result always extends, as we will see.

Lemma 8 *Let Γ be a complete, consistent theory containing Henkin witnesses. Let $\mathcal{M}$ a model of the rewrite rules, with underlying set the set of* *closed terms, and such that for any normal atom:*

$$\mathcal{M} \models A \text{ iff } \Gamma, A \not\vdash^{cf}_{\mathcal{R}}$$

then, for any proposition P:

$$\text{if } \Gamma, P \not\vdash^{cf}_{\mathcal{R}} \text{ then } \mathcal{M} \models P$$
$$\text{if } \Gamma \not\vdash^{cf}_{\mathcal{R}} P \text{ then } \mathcal{M} \not\models P$$

Proof: By induction on the well-founded order:

- P is a normal atom. This is the hypothesis on the model.
- $P = A \vee B$, case 3. Then $\Gamma, A \vee B \not\vdash^{cf}_{\mathcal{R}}$ implies:

$$\begin{cases} \Gamma, A \not\vdash^{cf}_{\mathcal{R}} \\ OR \\ \Gamma, B \not\vdash^{cf}_{\mathcal{R}} \end{cases}$$

  by induction hypothesis, that implies:

$$\begin{cases} \mathcal{M} \models A \\ OR \\ \mathcal{M} \models B \end{cases}$$

  i.e. $\mathcal{M} \models A \vee B$.

- $P = A \wedge B$ case 3, then $\Gamma, A \wedge B \not\vdash^{cf}_{\mathcal{R}}$ implies:

$$\begin{cases} \Gamma, A \not\vdash^{cf}_{\mathcal{R}} \\ AND \\ \Gamma, B \not\vdash^{cf}_{\mathcal{R}} \end{cases}$$

  by induction hypothesis:

$$\begin{cases} \mathcal{M} \models A \\ AND \\ \mathcal{M} \models B \end{cases}$$

  That result implies $\mathcal{M} \models A \wedge B$.

- $P = \neg Q$ case 3. That implies $\Gamma \not\vdash^{cf}_{\mathcal{R}} Q$, and then $\mathcal{M} \not\models Q$ by induction hypothesis, hence $\mathcal{M} \models P$.
- $P = \forall x Q$ case 3. $\Gamma, \forall x Q \not\vdash^{cf}_{\mathcal{R}}$ implies that for any closed term t, $\Gamma, A[t/x] \not\vdash^{cf}_{\mathcal{R}}$, by induction hypothesis, for any $t \in \mathcal{I}$, $\mathcal{M} \models Q[t/x]$, hence $\mathcal{M} \models P$.
- $P = \exists x Q$ case 3. With the help of Henkin witnesses, we can find a constant c such that $\Gamma, P[c/x] \not\vdash^{cf}_{\mathcal{R}}$, that implies by induction hypothesis that $\mathcal{M} \models Q[c/x]$, hence $\mathcal{M} \models \exists x P$.

- $P = \exists x Q$ case 3, that implies that for any closed term t, we have $\Gamma \not\vdash_{\mathcal{R}}^{cf} Q[t/x]$, and by induction hypothesis, for any $t \in \mathcal{I}$, $\mathcal{M} \not\models Q[t/x]$, that is to say $\mathcal{M} \not\models P$.

- $P = \forall x Q$ case 3, $\Gamma \not\vdash_{\mathcal{R}}^{cf} \forall x Q$ implies the existence of a constant c such that $\neg Q[c/x] \in \Gamma$. Hence, as Γ is consistent, we have $\Gamma, \neg Q[c/x] \not\vdash_{\mathcal{R}}^{cf}$. That implies $\Gamma \not\vdash_{\mathcal{R}}^{cf} Q[c/x]$ by lemma 6, and, by induction hypothesis, $\mathcal{M} \not\models Q[c/x]$, that is the same as $\mathcal{M} \not\models \forall x Q$.

- If P is a non normal atom, we rewrite it and we apply the induction hypothesis, because $\Gamma, P \not\vdash_{\mathcal{R}}^{cf}$ is equivalent to $\Gamma, P' \not\vdash_{\mathcal{R}}^{cf}$ by proposition 1, and because $\mathcal{M}$ is a model of te rewrite rules.

- the remaining cases are similar.

At each step, we are decreasing the order $\succ$. Because the order is well-founded, this induction terminates. □

7 Theorems

Theorem 9 (Completness) *Suppose we have a confluent rewrite system, a well-founded order compatible with the rewrite rules and having the sub-formula property.*

If $\mathcal{T}$ is a consistent denumerable theory, it has a model.

Proof: We construct a consistent, complete theory Γ containing $\mathcal{T}$ and having Henkin witnesses, like in section 6.1. Γ is expressed in a language $\mathcal{L}'$. Let P be a proposition such that $\mathcal{T} \vdash_{\mathcal{R}}^{cf} P$, and let $\mathcal{M}'$ the model constructed in section 6.2. By corollary 4 we have: $\mathcal{M}' \models \neg\mathcal{T}, P$.

Let Q be a proposition of $\mathcal{T}$. We have $\Gamma, Q \not\vdash_{\mathcal{R}}^{cf}$ by consistency of Γ, hence $\mathcal{M}' \models Q$ by lemma 8.

Thus we must have $\mathcal{M}' \models P$. If we call $\mathcal{M}$ the reducted model of $\mathcal{M}'$ to the language $\mathcal{L}$, we have $\mathcal{M} \models P$. □

Another formulation of this theorem is the following.

Proposition 10 *Let Γ be a theory and P be a proposition. If $\Gamma \models P$, then:*

$$\Gamma \vdash_{\mathcal{R}}^{cf} P$$

Proof: If $\Gamma \not\vdash_{\mathcal{R}}^{cf} P$, then by lemma 6 we have $\Gamma, \neg P \not\vdash_{\mathcal{R}}^{cf}$. Hence $\Gamma, \neg P$ has a model $\mathcal{M}$, according to theorem 9, and $\mathcal{M} \not\models \Delta$. □

We generalize this proposition to a set of propositions:

Proposition 11 *Let Γ be a theory and Δ be a multiset of propositions. If $\Gamma \models \Delta$ then:*

$$\Gamma \vdash_{\mathcal{R}}^{cf} \Delta$$

Proof: If $\Gamma \not\vdash^{cf}_{\mathcal{R}} \Delta$, by lemma 6, we have $\Gamma, \neg\Delta \not\vdash^{cf}_{\mathcal{R}}$. Hence, $\Gamma, \neg\Delta$ has a model, according to theorem 9. $\square$

And we conclude with the cut elimination theorem:

Theorem 12 (Cut elimination theorem) *If there exists a proof of:*

$$\Gamma \vdash_{\mathcal{R}} \Delta$$

there exists a proof of:

$$\Gamma \vdash^{cf}_{\mathcal{R}} \Delta$$

Proof: If $\Gamma \vdash_{\mathcal{R}} \Delta$ has a proof (possibly cuts), by soundness (lemma 3), we have $\Gamma \models \Delta$. We apply proposition 11 and obtain a proof of:

$$\Gamma \vdash^{cf}_{\mathcal{R}} \Delta$$

$\square$

8 Examples

Now, we give an example of sets of rewrite rules that have the cut elimination property. Theses rewrite systems are known as quantifier free rewrite systems [7].

Definition 11 *A rewrite system is said quantifier free if there is no rule $A \to P$ with P containing quantifiers.*

We suppose the rewrite system to be confluent, terminating and quantifier free.

We just have, for such a system, to construct a well-founded order. This is done in the following way.
We will write $\overline{P}$ the normal form of P, $\sharp_\forall(P)$ the number of quantifiers $(\forall, \exists)$ of the proposition P and $\sharp_\wedge(P)$ the number of connectors $(\wedge, \vee, \Rightarrow, \neg)$ of the proposition P.

Definition 12 (Order for normal propositions) *Let A, B be two normal propositions. We say that $A \succ B$:*

- *if $\sharp_\forall(A) > \sharp_\forall(B)$*
- *or if $\sharp_\forall(A) = \sharp_\forall(B)$ and $\sharp_\wedge(A) > \sharp_\wedge(B)$*

Definition 13 (Order for propositions) *Let A, B be two propositions. $A \succ B$ if $\overline{A} \succ \overline{B}$ or if $\overline{A} = \overline{B}$ and $A \to^{+}_{\mathcal{R}} B$.*

This order is well founded. Suppose we have an infinite sequence $A_1 \succ \ldots. \succ A_n$. As there is no infinite rewrite sequence $A_p \to^{+} \ldots \to^{+} A_q$, we can extract an infinite subsequence $A_{p(1)} \succ \ldots \succ A_{p(n)} \succ \ldots$ such that $\overline{A_{p(1)}} \succ \ldots \succ \overline{A_{p(n)}} \succ \ldots.$ But the order on normal propositions is the same as the order on the couple $(\sharp_\forall(P), \sharp_\wedge(P))$ that is well-founded. Hence, the extracted sequence is finite. This is a contradiction. Hence our order is well-founded.

This order is compatible with the rewrite rules. Let $A \rightarrow^+_{\mathcal{R}} B$. We have $\overline{A} = \overline{B}$ by termination and confluence. Hence, by the definition of the order, $A \succ B$.

This order has the subterm property (for any binary or unary connector). In fact, let $P = A\ \mathcal{C}\ B$ a proposition with a connector $\mathcal{C}$. We have $\overline{P} = \overline{A\ \mathcal{C}\ B} = \overline{A}\ \mathcal{C}\ \overline{B}$, because we have atomic rewrite rules. Hence, the number of quantifiers of $\overline{A}$ is at most the same that the one of $\overline{P}$, and the number of connectors strictly less. Thus $P \succ A$ and $P \succ B$.
Let now $P = \mathcal{Q}x\, A$ where $\mathcal{Q}$ is a quantifier. $\overline{A[t/x]}$ has stricly less quantifiers than $\overline{P}$ because the rewrite system is quantifier free. Hence $P \succ A[t/x]$ for any t.

We have constructed an order that fits to our hypotesis. Then, the cut elimination theorem holds for all theses systems. Many rules are quantifier free. For example, in set theory:

$$x \in A \cup B \rightarrow x \in A \vee x \in B$$
$$x \in A \cap B \rightarrow x \in A \wedge x \in B$$

Or in arithmetic:

$$\begin{aligned} x * y = 0 &\rightarrow x = 0 \vee y = 0 \\ x * 0 &\rightarrow 0 \end{aligned}$$

Conclusion

The cut elimination property is already proved for the systems of last section, but with the help of pre-models, a construction we replaced by models and an order condition. The next step would be to supress the order condition, that appears only in section 6.2, to have a more general condition on models. The difficult point would be then to construct an adequate model, maybe following the way of the pre-model construction [7].

The proof of [11] is an optimized proof for automatic proof searching. We proved in this paper that this result extends and can apply to proof theory. This paper also enhanced links between completness, models and cut-elimination, as seen in last section. It gives a new proof of cut-elimination for sequent calculus, that is the same as sequent calculus modulo without rewrite rules.

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