A Semantic Completeness Proof for Tableaux Modulo

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LPAR '06, November 14, 2006

Outline

Proving: Computation vs. Deduction

Computation

Deterministic search for a proof.

Deduction

Non-deterministic search for a proof.

Deduction modulo = computation + deduction

Deduction

Natural deduction, *sequent calculus*

Computation

- Term rewriting
$$S(x) + y \longrightarrow S(x + y)$$
- Equational axioms
$$x + y = y + x$$
- **Propositional rewriting**
$$x * y = 0 \longrightarrow x = 0 \vee y = 0$$

Extension of the sequent calculus

$\mathcal{R}$: set of propositional rewrite rules
$\mathcal{E}$: set of term rewrite rules and equational axioms.
$\equiv_{\mathcal{RE}}$: congruence defined by $\mathcal{R} \cup \mathcal{E}$.

Sequent modulo rule

$$\frac{\Gamma, A \vdash_{\mathcal{RE}} \Delta \qquad \Gamma, B \vdash_{\mathcal{RE}} \Delta}{\Gamma, P \vdash_{\mathcal{RE}} \Delta} \vee\text{-l}$$

if $P \equiv_{\mathcal{RE}} A \vee B$.

Benefits

- Shorter more readable proofs

$$\frac{}{\Gamma, 4 = 4 \vdash_{\mathcal{RE}} 2 + 2 = 4, \Delta} \text{ axiom, } \mathcal{RE} \hat{=} \text{ Peano's arithmetic}$$

- Powerful general framework for theories without axioms
  - HOL (Dowek, Hardin, Kirchner, MSCS 01)
  - (Heyting's) arithmetic (Dowek, Werner, RTA '05)
  - Zermelo's set theory (Dowek, Miquel)

Tableaux modulo: what we want

- Better free variable tableaux for deduction modulo (Bonichon, IJCAR'04)
- Term rewriting and propositional rewriting in $\mathcal{R}$
- Soundness (semantic)
- Completeness (semantic)

Expanding tableaux

- Free-variable constrained tableaux of constrained formulas
- Rules for:
  - Connectors: α and β rules
  - Quantifiers: γ and δ rules
  - Rewriting (on terms and propositions)
  - Closure

Rules

$$\frac{\alpha(A, B)}{A, B}$$

$$\frac{\alpha(A, B)}{\begin{array}{c} A \wedge B \\ \neg(A \vee B) \\ \neg(A \Rightarrow B) \end{array}}$$

Rules

$$\frac{\alpha(A,B)}{A, B} \qquad \frac{\beta(A,B)}{A \mid B}$$

$$\frac{\beta(A,B)}{\begin{array}{c} A \vee B \\ \neg(A \wedge B) \\ A \Rightarrow B \end{array}}$$

Rules

$$\frac{\alpha(A, B)}{A, B} \qquad \frac{\beta(A, B)}{A \mid B}$$

$$\frac{\gamma(x, A)}{A[x := X]}$$

$$\frac{\gamma(x, A)}{\begin{array}{c}\forall x A \\ \neg(\exists x A)\end{array}}$$

Rules

$$\frac{\alpha(A, B)}{A, B} \qquad \frac{\beta(A, B)}{A \mid B}$$

$$\frac{\gamma(x, A)}{A[x := X]} \qquad \frac{\delta(x, A)}{A[x := f_{\text{sko}}(\text{args})]}$$

$$\frac{\delta(x, A)}{\begin{array}{c}\exists x A \\ \neg(\forall x A)\end{array}}$$

Rewriting

$\mathcal{T} \cdot \mathcal{C}$: tableau $\mathcal{T}$ with global constraint $\mathcal{C}$
P_c: proposition P with local constraint c

$$\frac{\Gamma_1, P_c \mid \ldots \mid \Gamma_n \cdot \mathcal{C}}{\Gamma_1, P_c, P_{c'}[r]_\omega \mid \ldots \mid \Gamma_n \cdot \mathcal{C}}\ \texttt{rw}$$

if $l \rightarrow r$, and $c' = c \cup \{P_{|\omega} \stackrel{?}{=} l\}$

Soundness

Theorem (Soundness)

If Γ *has a* $\mathcal{R}$*-model, then we can not derive the closed tableau* $\odot$ *from* Γ *using the rules for tableaux modulo.*

Proof.

As usual. $\mathcal{R}$-models are extensions of boolean models such that for any propositions $P \equiv_{\mathcal{R}} Q$, $|P| = |Q|$. $\square$

Systematic tableau expansion

$\forall x P(x) \wedge (\neg P(c) \vee \neg Q(a))$

$\mathcal{R} = P(a) \rightarrow Q(a)$

Rule:

$\mathcal{C} = \emptyset$

Systematic tableau expansion

$\mathcal{R} = P(a) \rightarrow Q(a)$

Rule: $\dfrac{\alpha(A, B)}{A, B}$

$\mathcal{C} = \emptyset$

$$\begin{array}{c} \forall x P(x) \wedge (\neg P(c) \vee \neg Q(a)) \\ | \\ \forall x P(x) \\ | \\ \neg P(c) \vee \neg Q(a) \end{array}$$

Systematic tableau expansion

$$\forall x P(x) \wedge (\neg P(c) \vee \neg Q(a))$$

$$\forall x P(x)$$

$$\neg P(c) \vee \neg Q(a)$$

$$\neg P(c) \qquad \neg Q(a)$$

$\mathcal{R} = P(a) \to Q(a)$

Rule: $\dfrac{\beta(A, B)}{A \mid B}$

$\mathcal{C} = \emptyset$

	l	r
lgth	4	4
$\gamma >$ rw	1	1

Systematic tableau expansion

$$\forall x P(x) \wedge (\neg P(c) \vee \neg Q(a))$$

$$\forall x P(x)$$

$$\neg P(c) \vee \neg Q(a)$$

$$\neg P(c) \qquad \neg Q(a)$$

$$P(X)$$

$\mathcal{R} = P(a) \rightarrow Q(a)$

Rule: $\dfrac{\gamma(x, A)}{A[x := X]}$

$\mathcal{C} = \emptyset$

	l	r
lgth	5	4
$\gamma >$ rw	0	1

Systematic tableau expansion

$\forall x P(x) \land (\neg P(c) \lor \neg Q(a))$

$\forall x P(x)$

$\neg P(c) \lor \neg Q(a)$

$\neg P(c)$ $\quad$ $\neg Q(a)$

$P(X)$ $\quad$ $P(Y)$

$\mathcal{R} = P(a) \to Q(a)$

Rule: $\dfrac{\gamma(x, A)}{A[x := X]}$

$\mathcal{C} = \emptyset$

	l	r
lgth	5	5
$\gamma >$ rw	0	0

Systematic tableau expansion

$$\forall x P(x) \wedge (\neg P(c) \vee \neg Q(a))$$

$$\forall x P(x)$$

$$\neg P(c) \vee \neg Q(a)$$

$$\neg P(c) \qquad \neg Q(a)$$

$$P(X) \qquad P(Y)$$

$$Q(a)_{X \stackrel{?}{=} a}$$

$\mathcal{R} = P(a) \rightarrow Q(a)$

Rule: $\dfrac{P(l)}{P(r)}$

$\mathcal{C} = \emptyset$

	l	r
lgth	6	5
$\gamma >$ rw	1	0

Systematic tableau expansion

$$\forall x P(x) \wedge (\neg P(c) \vee \neg Q(a))$$

$$\forall x P(x)$$

$$\neg P(c) \vee \neg Q(a)$$

$$\neg P(c) \qquad \neg Q(a)$$

$$P(X) \qquad P(Y)$$

$$Q(a)_{X \stackrel{?}{=} a} \qquad Q(a)_{Y \stackrel{?}{=} a}$$

$\mathcal{R} = P(a) \rightarrow Q(a)$

Rule: $\dfrac{P(l)}{P(r)}$

$\mathcal{C} = \emptyset$

	l	r
lgth	6	6
$\gamma >$ rw	1	1

Systematic tableau expansion

$$\forall x P(x) \wedge (\neg P(c) \vee \neg Q(a))$$
$$\forall x P(x)$$
$$\neg P(c) \vee \neg Q(a)$$

$\neg P(c)$	$\neg Q(a)$
$P(X)$	$P(Y)$
$Q(a)_{X \stackrel{?}{=} a}$	$Q(a)_{Y \stackrel{?}{=} a}$
$\odot$	$\odot$

$\mathcal{R} = P(a) \rightarrow Q(a)$

Rule: closure

$\mathcal{C} = \{X \stackrel{?}{=} c, Y \stackrel{?}{=} a\}$

	l	r
lgth	6	6
$\gamma >$ rw	1	1

Proving completeness

Theorem (Completeness)

Let $\mathcal{R}$ be a confluent terminating rewrite system, and Γ a set of propositions. If the systematic tableau procedure rooted at Γ has an open branch, then Γ has a $\mathcal{R}$-model.

Usual model construction

From a complete open branch:

- Semi-valuation (Hintikka set)
- Model

And that's it !

Is that enough?

- Models must be models of the rewrite rules too!
- The intended meaning of $P \rightarrow Q$ is $P \Leftrightarrow Q$
- But we begin by defining the usual semi-valuation V from the open branch ...

Some problems

$\mathcal{R}$-semi-valuation ($\mathcal{V}$)

When $P \longrightarrow Q$ (using $l \longrightarrow r$), $V(P)$ must be defined when $V(Q)$ is.

$\mathcal{R}$-partial valuation ($\tilde{V}$)

We would like to have $\mathcal{V}(\forall x(A(x) \wedge B(x))) = 1$ when for every term t we have $\mathcal{V}(A(t)) = 1$ and $\mathcal{V}(B(t)) = 1$.

Confluence and termination of $\mathcal{R}$

Not enough for completeness (Dowek-Werner, Hermant) !

Positivity

Condition

Propositional rewrite rules $l \rightarrow r \in \mathcal{R}$ are such that all atoms occurring in r occur positively

Model construction

- if A is an atom, and if $\tilde{V}(A)$ is defined, set $|A|_{\mathcal{R}} = \tilde{V}(A)$.
- if A is an atom, and if $\tilde{V}(A)$ is not defined, set $|A|_{\mathcal{R}} = 1$.
- if P is a compound proposition, set $|P|_{\mathcal{R}}$ accordingly.

Order

Condition

We consider a confluent rewrite system and a well-founded order $\prec$ such that:

- if $P \longrightarrow Q$ then $Q \prec P$.
- if A is a subformula of B then $A \prec B$.

Model construction

- if A is a normal atom, set $|A|_{\mathcal{R}} = \tilde{V}(A)$, if defined. Else set $|A|_{\mathcal{R}}$ arbitrarily.
- if A is not a normal atom, set $|A|_{\mathcal{R}} = |A\downarrow|_{\mathcal{R}}$.
- if P is a compound proposition, set $|P|_{\mathcal{R}}$ from the interpretation of its immediate subformulas.

Mixed

Condition

Consider two rewrite systems $\mathcal{R}_>$ and $\mathcal{R}_+$.
$\mathcal{R} = \mathcal{R}_> \cup \mathcal{R}_+$ must be confluent and terminating.
$\mathcal{R}_+$ is right-normal for $\mathcal{R}_>$

Right-normality

Let two rewrite systems $\mathcal{R}'$ and $\mathcal{R}$. $\mathcal{R}'$ is right normal for $\mathcal{R}$ if, for any propositional rule $l \to r \in \mathcal{R}'$, all the instances of atoms of r by $\mathcal{R}$-normal substitutions σ are in normal form for $\mathcal{R}$.

What we have

- Correct and complete tableau method for deduction modulo
- Somewhat better than the previous one
- Extract classes of rewrite systems

Cut elimination

- Not always normalization

$$R \in R \longrightarrow \forall y\ (\forall x(y \in x \Rightarrow R \in x) \Rightarrow (y \in R \Rightarrow (A \Rightarrow A)))$$

- Semantic cut elimination (admissibility)

Cut elimination

$$\begin{tikzcd}
\Gamma \vdash \Delta \arrow[r, shift left, "\text{correction}"] \arrow[d, "\begin{array}{c}\text{Gentzen}\\ \text{Tait-Girard}\\ \text{Dowek-Werner}\\ \dots\end{array}"'] & \Gamma \models \Delta \arrow[l, shift left, "\text{completeness}"] \\
\Gamma \vdash^{cf} \Delta &
\end{tikzcd}$$

Cut elimination

$$\begin{tikzcd}
\Gamma \vdash \Delta \arrow[rr, shift left, "\text{correction}"] \arrow[d, "\begin{array}{c}\text{Gentzen}\\ \text{Tait-Girard}\\ \text{Dowek-Werner}\\ \ldots\end{array}"'] & & \Gamma \models \Delta \arrow[ll, shift left, "\text{completeness}"] \arrow[dll, "\text{strong completeness}"] \\
\Gamma \vdash^{cf} \Delta & &
\end{tikzcd}$$

Cut elimination

$$\begin{tikzcd}
\Gamma \vdash \Delta \arrow[r, "\text{correction}"] & \Gamma \models \Delta \arrow[dl, "\text{strong completeness}"] \\
\Gamma \vdash^{cf} \Delta &
\end{tikzcd}$$

Cut elimination

$$
\begin{tikzcd}
\Gamma \vdash \Delta \arrow[rr, "\text{soundness}"] & & \Gamma \models \Delta \arrow[d, "\text{completeness}"] \\
\Gamma \vdash^{cf}_{\mathcal{R}} \Delta & & \mathrm{Tab}(\Gamma, \neg\Delta) \hookrightarrow \odot \arrow[ll, "\text{soundness}"']
\end{tikzcd}
$$

Further research

- Companion paper on intuitionistic logic: constructive cut elimination
- More classes for semantic completeness
- Using semantic methods for HOL-C (with combinators)
- Ongoing implementation in Zenon prover (Doligez)