Models for Normalization(s)

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11 Mars 2008

Natural Deduction: the logical framework

- first-order logic: function and predicate symbols, logical connectors: $\wedge, \vee, \Rightarrow, \neg$, and quantifiers $\forall, \exists$.

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\begin{gathered}
Even(0) \\
\forall n (Even(n) \Rightarrow Odd(n+1)) \\
\forall n (Odd(n) \Rightarrow Even(n+1))
\end{gathered}
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\end{gathered}
$$

- a sequent :

$$
\overbrace{\Gamma}^{\text{hyp.}} \vdash \overbrace{A}^{\text{conc.}}
$$

- rules to form them: natural deduction (or sequent calculus)
- framework: intuitionnistic logic (classical, linear, higher-order, constraints ...)

Deduction System : natural deduction (NJ)

- A deduction rule:

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

- introduction and elimination rules

$$\frac{}{\Gamma, A \vdash A}\text{axiom}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge\text{-i} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge\text{-e1} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge\text{-e2}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow\text{-i} \qquad \frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B} \Rightarrow\text{-e}$$

$$\frac{\Gamma \vdash \forall x A[x]}{\Gamma \vdash A[t]} \forall\text{-e, any } t \qquad \frac{\Gamma \vdash A[x]}{\Gamma \vdash \forall x A[x]} \forall\text{-i, } x \text{ free}$$

Example: 1

$$\forall x P(x) \vdash P(0) \land P(1)$$

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$$\dfrac{\forall x P(x) \vdash P(0) \qquad \forall x P(x) \vdash P(1)}{\forall x P(x) \vdash P(0) \wedge P(1)} \wedge\text{-i}$$

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Deduction modulo: allowed rewriting

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$$l \rightarrow r$$

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$$x + S(y) \rightarrow S(x + y)$$

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$$x * y = 0 \to x = 0 \vee y = 0$$

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- advantage: expressiveness
- we obtain a congruence modulo $\mathcal{R}$ (chosen set of rules): $\equiv$
- deduction rules transformation:

$$\text{axiom}\ \frac{}{\Gamma, A \vdash A} \qquad \text{becomes} \qquad \frac{}{\Gamma, A \vdash B}\ \text{axiom},\ A \equiv B$$

Deduction modulo : natural deduction modulo - first presentation

$$\frac{}{\Gamma, A \vdash A}\text{axiom}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge\text{-i} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge\text{-e1} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge\text{-e2}$$

$$\Rightarrow\text{-i}\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \qquad \frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B} \Rightarrow\text{-e}$$

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Deduction modulo : first presentation

Add then the following conversion rule:

$$\frac{\Gamma \vdash A}{\Gamma \vdash B}\ A \equiv B$$

Deduction modulo : natural deduction modulo, reloaded

$$\frac{}{\Gamma, A \vdash B}\text{axiom, } A \equiv B$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash C} \wedge\text{-i, } C \equiv A \wedge B \qquad \frac{\Gamma \vdash C}{\Gamma \vdash A} \wedge\text{-e1,} C \equiv A \wedge B \qquad \frac{\Gamma \vdash C}{\Gamma \vdash B} \wedge\text{-e2,} C \equiv A \wedge B$$

$$\Rightarrow\text{-i, } C \equiv A \wedge B \; \frac{\Gamma, A \vdash B}{\Gamma \vdash C} \qquad \frac{\Gamma \vdash C \qquad \Gamma \vdash A}{\Gamma \vdash B} \Rightarrow\text{-e, } C \equiv A \wedge B$$

$$\frac{\Gamma \vdash A[x]}{\Gamma \vdash B}\forall\text{-i, } x \text{ free,} B \equiv \forall x A[x] \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A[t]}\forall\text{-e, any } t, B \equiv \forall x A[x]$$

Example: 3

- consider the rewriting system $\mathcal{R}$:

$$\begin{array}{rcl} P(0) & \rightarrow & A \\ P(1) & \rightarrow & B \end{array}$$

$$\forall x P(x) \vdash A \wedge B$$

Example: 3

- consider the rewriting system $\mathcal{R}$:

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A Cut: a detour

$$\dfrac{\Gamma \vdash A \qquad \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\Rightarrow\text{-i}}{\Gamma \vdash B}\Rightarrow\text{-e}$$

- show $\Gamma \vdash A$ and $\Gamma, A \vdash B$
- then, you have showed $\Gamma \vdash B$
- it is the application of a lemma.

A cut: a detour

$$\cfrac{\cfrac{\cfrac{\pi_1}{\Gamma \vdash A} \qquad \cfrac{\pi_2}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}\ \wedge\text{-i}}{\Gamma \vdash A}\ \wedge\text{-e}$$

General pattern of a cut: an introduction rule, followed by an elimination on the same symbol.
This is unnecessary, consider only π_1.

$$\frac{\pi_1}{\Gamma \vdash A}$$

A cut: a detour

And in the other proof:

$$\dfrac{\dfrac{\theta}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\pi}{\Gamma, A \vdash B}}{\Gamma \vdash A \Rightarrow B}\Rightarrow\text{-i}}{\Gamma \vdash B}\Rightarrow\text{-e}$$

Look in π what is happening:

$$\dfrac{\text{axiom}\ \dfrac{}{\Gamma, A, \Delta \vdash C_1} \qquad \dfrac{\text{axiom}}{\Gamma, A, \Delta \vdash C_i} \qquad \dfrac{}{\Gamma, A, \Delta \vdash C_n}\ \text{axiom}}{\dfrac{[\text{NJ rules}]}{\Gamma, A \vdash B}}$$

Now, assume $C_1 = A$ (and no other C_i is).

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$$\dfrac{\dfrac{\theta}{\Gamma, \Delta \vdash C_1} \qquad \dfrac{\text{axiom}}{\Gamma, \Delta \vdash C_i} \qquad \dfrac{}{\Gamma, \Delta \vdash C_n}\text{axiom}}{\dfrac{[\text{NJ rules}]}{\Gamma \vdash B}}$$

Now, assume $C_1 = A$ (and no other C_i is). We eliminated A from the hypothesis. π is directly a proof of $\Gamma \vdash B$ *replace uses of* A *(nb: axioms) by* θ. In clear: don't use the lemma, reprove its instances.

A cut: a detour

In deduction modulo:

$$\dfrac{\dfrac{\theta}{\Gamma \vdash A'} \qquad \dfrac{\dfrac{\pi}{\Gamma, A \vdash B}}{\Gamma \vdash C} \Rightarrow\text{-i}, C \equiv A \Rightarrow B}{\Gamma \vdash B'} \Rightarrow\text{-e}, C \equiv A' \Rightarrow B'$$

- need for cut elimination: the heart of logic.

A cut: a detour

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- need for cut elimination: the heart of logic.
- two main methods:
  - semantic: cut admissibility.
  - syntactic: proof normalization.

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- need for cut elimination: the heart of logic.
- two main methods:
  - semantic: cut admissibility.
  - syntactic: proof normalization.
- indecidable, need for conditions on $\mathcal{R}$.

The semantical method

$$
\begin{tikzcd}
\Gamma \vdash A \arrow[rrr, shift left, "\text{soundness}"] \arrow[d, "\begin{array}{c}\text{Gentzen}\\\text{Tait-Girard}\\\text{Dowek-Werner}\\\text{...}\end{array}"'] & & & \Gamma \models A \arrow[lll, shift left, "\text{completeness}"] \\
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$$

Heyting algebras

- a universe Ω
- an order

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- an order
- operations on it: lowest upper bound (join: $\cup$), greatest lower bound (meet: $\cap$).

$$a \cap b \leq a \quad a \cap b \leq b \quad c \leq a \text{ and } c \leq b \text{ implies } c \leq a \cap b$$
$$a \leq a \cup b \quad b \leq a \cup b \quad a \leq c \text{ and } b \leq c \text{ implies } a \cup b \leq c$$

- like Boolean algebras, with weaker complement

an example

- $\mathbb{R}$ and open sets (infinite g.l.b. is not infinite intersection)

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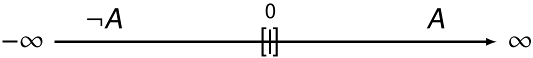
- $\mathbb{R}$ and open sets (infinite g.l.b. is not infinite intersection)
- complement is weaker:



A model

- a domain $\mathcal{D}$ to interpret the first-order terms.
- a Heyting algebra
- a interpretation function for each symbol:

$$\begin{array}{rcl} \hat{f} : \mathcal{D}^n & \rightarrow & \mathcal{D} \\ \hat{P} : \mathcal{D}^m & \rightarrow & \mathcal{D} \end{array}$$

- that we extend readily to all terms and all formulae and terms:

$$\begin{array}{rcl} (x)^*_\phi & := & \phi(x) \\ (f(t_1, \cdots, t_n))^*_\phi & := & \hat{f}(((t_1)^*_\phi, \cdots, (t_n)^*_\phi)) \\ (P(t_1, \cdots, t_n))^*_\phi & := & \hat{P}(((t_1)^*_\phi, \cdots, (t_n)^*_\phi)) \\ (A \wedge B)^*_\phi & := & (A)^*_\phi \cap (B)^*_\phi \end{array}$$

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- degree of freedom: how to choose $\hat{f}$ and $\hat{P}$.
- in deduction modulo, additional condition:

$$A \equiv_{\mathcal{R}} B \text{ implies } A^* = B^*$$

Cannonical model: Lindenbaum algebra

- defined for provability
- elements of Ω: the equivalence class of formulae $[A]$.

$$[A] := \{B \mid \vdash A \Leftrightarrow B\}$$

- meet: $[A] \cap [B]$ iff $[A \wedge B]$
- order: $[A] \leq [B]$ iff $\vdash A \Rightarrow B$

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- with this model, one proves completeness

Cannonical model: Lindenbaum algebra

- defined for provability with cuts
- elements of Ω: the equivalence class of formulae $[A]$.

$$[A] := \{B \mid \vdash A \Leftrightarrow B\}$$

- "intersection": $[A] \cap [B]$ iff $[A \wedge B]$
- "order": $[A] \leq [B]$ iff $\vdash A \Rightarrow B$
- and so on ... (domain $\mathcal{D}$: open terms)
- with this model, one proves completeness: cuts are needed for transitivity of the order.

Cut-free cannonical model

- defined for provability without cuts
- elements of Ω: the contexts proving A cut-free.

$$[A] := \{\Gamma \mid \Gamma \vdash^* A\}$$

- the $[A]$ are the basis. Saturate then Ω with their (arbitrary) intersection and pseudo-union (l.u.b.):

$$a \cup b = \bigcap \{[A] \mid a \subseteq [A] \text{ and } b \subseteq [A]\}$$

- order: $a \leq b$ iff $a \subseteq b$
- and so on ...
- with this model, one proves cut-free completeness.

Deduction modulo

- what about the domain ?
- what about the validity of the rewrite rules ?

$$A \equiv_{\mathcal{R}} B \text{ implies } A^* = B^*$$

Deduction modulo

- what about the domain: it depends on $\mathcal{R}$ - usually the open term is sufficient.
- what about the validity of the rewrite rules: choose carefully the interpretation of predicates and function symbols, depends on $\mathcal{R}$.

An example: Simple Theory of Types

- aka higher-order (intuitionistic) logic.
- basic types o, ι, and arrow: $o \to o$, $o \to \iota$, ...
- constants of each type
- application $(t\ u)$ and λ-abstraction or combinators: S, K
- logical connectors: constants $\wedge : o \to o \to o$, ...
- e.g. we can form the formula: $\forall P.P$

cut admissibility in STT (no modulo)

- problem number one, circularity:

$$\dfrac{\dfrac{\vdots}{\vdash (\mathfrak{P} \Rightarrow \mathfrak{P})}}{\vdash \forall.P(P \Rightarrow P)}$$

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$$\dfrac{\dfrac{\vdots}{\vdash (\forall P.(P \Rightarrow P) \Rightarrow \forall P.(P \Rightarrow P))}}{\vdash \forall .P(P \Rightarrow P)}$$

- no more induction on the size of the formulae.

cut admissibility in STT (no modulo)

- problem number one, circularity:
- no more induction on the size of the formulae.
- solution, same as slide 32 of Dowek:

Define R_A: quantify over all R_B: Circular

Avoid circularity: define C a priori, quantify over C instead,
Prove a posteriori that $R_B \in C$.

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$$\{\alpha \in \Omega \mid A \in \alpha \subseteq [A]\}$$

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- define "semantic candidates" [Okada] for $(A)^*$ without induction:

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- then quantify over all truth-values candidates. Identifies which of the α is $(A)^*$.

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- interpret everything within those domains, e.g.:

$$\hat{\wedge} := \langle \wedge, \lambda\langle B, b\rangle.\langle \wedge \cdot B, \lambda\langle C, c\rangle.\langle \wedge \cdot B \cdot C, b \cap c\rangle\rangle\rangle$$

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- then, "extract" the truth value:

$$\omega(A^*) = \pi_2(A^*)$$

STT in deduction modulo

- same types, same symbols $\dot{\wedge}, \dot{\forall}, \cdots$
- application:

$$
\begin{array}{rcl}
K \cdot x \cdot y & \rightarrow & x \\
S \cdot x \cdot y \cdot z & \rightarrow & (xz)(yz)
\end{array}
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- how to express $\forall P.P$ in a first-order setting ?

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- solution: embed P into $\varepsilon(P)$, and define:

$$\begin{array}{rcl} \varepsilon(\dot{\wedge} \cdot A \cdot B) & \rightarrow & \varepsilon(A) \wedge \varepsilon(B) \\ \varepsilon(\dot{\forall} A) & \rightarrow & \forall x.\varepsilon(Ax) \end{array}$$

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- duplication of “connectors”: $\dot{\wedge}$ (of the type hierarchy) connecting terms and $\wedge$, connecting propositions.
- two “formulae”: P, a term, and $\varepsilon(P)$, at the logical level.
- ε is the only predicate symbol.

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\varepsilon(\dot{\wedge} \cdot A \cdot B) & \rightarrow & \varepsilon(A) \wedge \varepsilon(B) \\
\varepsilon(\dot{\forall} A) & \rightarrow & \forall x.\varepsilon(Ax)
\end{array}
$$

- duplication of “connectors”: $\dot{\wedge}$ (of the type hierarchy) connecting terms and $\wedge$, connecting propositions.
- two “formulae”: P, a term, and $\varepsilon(P)$, at the logical level.
- ε is the only predicate symbol.
- ε embeds in the syntax the ω is in the semantics: separates truth value and propositional content.

The normalization method(s)

- Curry-Howard: proofs = programs
- formulas = types
- proof tree = typing tree
- at the heart of proof assistants (PVS, Coq, Isabelle, ...)
- when a program calculates, it performs a cut elimination procedure.

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- at the heart of proof assistants (PVS, Coq, Isabelle, ...)
- when a program calculates, it performs a cut elimination procedure.
- show that all typables function terminates.

Curry-Howard correspondence

- Notation for proofs. Give a name to each of the hypothesis:

$$\Gamma = x_1 : A_1, \ldots, x_n : A_n$$

$$\frac{}{\Gamma, x : A \vdash x : A}\ \text{Axiom} \qquad \frac{\Gamma \vdash \pi : A \wedge B}{\Gamma \vdash \mathit{fst}(\pi) : A}\ \wedge\text{-e1}$$

$$\frac{\Gamma \vdash \pi_1 : A \qquad \Gamma \vdash \pi_2 : B}{\Gamma \vdash \langle \pi_1, \pi_2 \rangle : A \wedge B}\ \wedge\text{-i} \qquad \frac{\Gamma \vdash \pi : A \wedge B}{\Gamma \vdash \mathit{snd}(\pi) : A}\ \wedge\text{-e2}$$

$$\frac{\Gamma, x : A \vdash \pi : B}{\Gamma \vdash \lambda x.\pi : A \Rightarrow B}\ \Rightarrow\text{-i} \qquad \frac{\Gamma \vdash \pi' : A \qquad \Gamma \vdash \pi : A \Rightarrow B}{\Gamma \vdash (\pi \pi') : B}$$

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- **very** similar to a type system
- in deduction modulo, rewrite rules are silent:

$$\frac{\Gamma \vdash \pi : A}{\Gamma \vdash \pi : B}\ A \equiv B$$

Cut elimination with proof terms

- Cut elimination is a **process**, similar to function execution.

$$\dfrac{\dfrac{\Gamma \vdash \pi_1 : A \qquad \Gamma \vdash \pi_2 : B}{\Gamma \vdash \langle \pi_1, \pi_2 \rangle : A \wedge B}\,\wedge\text{-i}}{\Gamma \vdash \mathit{fst}(\langle \pi_1, \pi_2 \rangle) : A}\,\wedge\text{-e} \quad \rhd \quad \Gamma \vdash \pi_1 : A$$

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- showing that every proof normalizes: the cut elimination process terminates.

Normalization [Dowek,Werner]

- deduction modulo is high-level: reducibility candidates.

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- to each formula A, associates a candidate $[\![A]\!]$. Show that if $\Gamma \vdash \pi : A$ then $\pi \in [\![A]\!]$.
- in deduction modulo, if $A \equiv B$, additional constraint:

$$[\![A]\!] = [\![B]\!]$$

Towards “usual” semantics

- such methods are defined in deduction modulo (Heyting arithmetic, higher-order logic, Zermelo’s set theory, ...)

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- such methods are defined in deduction modulo (Heyting arithmetic, higher-order logic, Zermelo’s set theory, ...)
- the sets of candidates have a structure: pseudo Heyting algebras, or truth value algebras (TVA) [Dowek].

Heyting algebras

- a universe Ω
- an order
- operations on it: lowest upper bound (join: $\cup$), greatest lower bound (meet: $\cap$ – intersection).

$$a \cap b \leq a \quad a \cap b \leq b \quad c \leq a \text{ and } c \leq b \text{ implies } c \leq a \cap b$$
$$a \leq a \cup b \quad b \leq a \cup b \quad a \leq c \text{ and } b \leq c \text{ implies } a \cup b \leq c$$

- like Boolean algebras, with weaker complement

pseudo-Heyting algebras, aka Truth Values Algebras

- a universe Ω
- a pre-order: $a \leq b$ and $b \leq a$ with $a \neq b$ possible.
- operations on it: lowest upper bound (join: $\cup$ – pseudo union), greatest lower bound (meet: $\cap$ – intersection).

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Candidates form a pseudo-Heyting algebra

- $\top = \bot = \mathcal{SN}$
- $[\![A]\!] \cap [\![B]\!] = [\![A \wedge B]\!]$
- and so on.
- pre-order: trivial one.
- But $[\![A \wedge A]\!] \leq\geq [\![A]\!]$ only.

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- consistency: there exists a model.
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Super consistency

- consistency: there exists a model.
- super-consistency: for every TVA, there exists a model (interpretation): construction has to be uniform.
- condition in DM: $A \equiv B$ implies $[\![A]\!] = [\![B]\!]$

Super consistency

- e.g. higher-order logic is super-consistent:

$$
\begin{array}{rcl}
M_\iota & = & \iota \text{ (dummy)} \\
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- hence, it has a model in the pseudo-Heying Algebra of candidates
- $\Gamma \vdash \pi : A$ implies $\pi \in [\![A]\!]$.
- the system enjoys proof normalization.

Towards usual semantics

$$\begin{tikzcd}
\Gamma \vdash A \arrow[rr, "\text{soundness}"] \arrow[d, "\begin{array}{c}\text{Gentzen}\\ \text{Tait-Girard}\\ \text{Dowek-Werner}\\ \ldots\end{array}"'] & & \Gamma \models A \arrow[dll, "\text{strong completeness}"] \\
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$$[\![A]\!] = \{\Gamma \vdash B \text{such that ...}\}$$

Towards usual semantics

- How to transform a TVA into a Heyting algebra.
- assume we have a model $\mathcal{M}$, $[\![\_]\!]$ in the previous pseudo-Heyting algebra of sequents.
- first idea: pseudo-Heyting to Heyting by quotienting.

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- How to transform a TVA into a Heyting algebra.
- assume we have a model $\mathcal{M}, [\![\_]\!]$ in the previous pseudo-Heyting algebra of sequents.
- first idea: pseudo-Heyting to Heyting by quotienting.
- trivial pseudo order implies $\top = \bot$.

The link: fibering

define

$$[A]^{\sigma}_{\phi} = \{B_1, \cdots, B_n \mid \forall \Delta,\ \text{if}\ \Delta \vdash B_i \in [\![B_i]\!],\ \text{then}\ \Delta \vdash A \in [\![A]\!]_{\phi}\}$$

- $[\![A]\!]_{\phi}$ contains sequents $\Delta \vdash B$. Extract the contexts corresponding to A.

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- this forms a Heyting algebra ($[A]$: basis)
- interpretation of formulas in it:

$$A^* = [A]^{\sigma}_{\phi}$$

Wait a minute !

- interpretation ? $[A]_{\phi}^{\sigma}$.

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- Need for *one single* substitution. hybridization: $\sigma \times \phi$.

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- interpretation for symbols

$$\begin{aligned}
\hat{f}^{\mathcal{D}}(\langle t_1, d_1\rangle, ..., \langle t_n, d_n\rangle) &= \langle f(t_1, ..., t_n), \hat{f}^{\mathcal{M}}(d_1, ..., d_n)\rangle \\
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$$\langle t, v\rangle \odot \langle t', v'\rangle = \langle (tt'), (vv')\rangle$$

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- proof resembles the proof for normalization.

Cut admissibility

Assume $\Gamma \vdash A$ has a proof (with cuts)

- $[\Gamma] \leq [A]$ in $\mathcal{D}$ by (usual) soundness
- $\Gamma \in [\Gamma]$
- $\Gamma \in [A]$ implies $\Gamma \vdash_{cf} A$
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- Q.E.D.
- compared to: $\Gamma \vdash \pi : A$ implies $\pi \in [\![A]\!]$, and hence π is $\mathcal{SN}$.

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\end{tikzcd}
$$

- This diagram does not commute in deduction modulo.

Further work

- what is the computational content of this algorithm ?
- there is normalization by evaluation work, but in a Kripke style: links ?
- do the proof terms (candidates) always have a “pseudo-” structure ?
- realizing rewrite rule not with $\lambda x.x$ (not silently), could recover (some) normalization and make the previous diagram commute again.

$$\frac{\Gamma \vdash \pi : A}{\Gamma \vdash \pi : B} \; A \equiv B$$