Models for Normalization(s)

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Deduction and Computation

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- It has been forgotten by the formalization of the mathematics.
- reborn with informatics: rewriting rules.
- we need a balance between deduction steps and computation steps.

Natural Deduction: the logical framework

- first-order logic: function and predicate symbols, logical connectors: $\wedge, \vee, \Rightarrow, \neg$, and quantifiers $\forall, \exists$.

$$Even(0)$$
$$\forall n(Even(n) \Rightarrow Odd(n+1))$$
$$\forall n(Odd(n) \Rightarrow Even(n+1))$$

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- a sequent :

$$\overbrace{\Gamma}^{\text{hyp.}} \vdash \overbrace{A}^{\text{conc.}}$$

- rules to form them: natural deduction (or sequent calculus)
- framework: intuitionnistic logic (classical, linear, higher-order, constraints ...)

Deduction System : natural deduction (NJ)

- A deduction rule:

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

- introduction and elimination rules

$$\frac{}{\Gamma, A \vdash A}\text{axiom}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge\text{-i} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge\text{-e1} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge\text{-e2}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow\text{-i} \qquad \frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B} \Rightarrow\text{-e}$$

$$\frac{\Gamma \vdash \forall x A[x]}{\Gamma \vdash A[t]}\forall\text{-e, any } t \qquad \frac{\Gamma \vdash A[x]}{\Gamma \vdash \forall x A[x]}\forall\text{-i, } x \text{ free}$$

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$$

Axioms vs. rewriting

Axioms	Rewriting
$x + S(y) = S(x + y)$ $x + 0 = x$ $x * 0 = 0$ $x * S(y) = x + x * y$ $(x * y = 0) \Leftrightarrow (x = 0 \vee y = 0)$	$x + S(y) \rightarrow S(x + y)$ $x + 0 \rightarrow x$ $x * 0 \rightarrow 0$ $x * S(y) \rightarrow x + x * y$ $(x * y = 0) \rightarrow (x = 0 \vee y = 0)$
$\dfrac{\dfrac{\vdots}{\mathcal{T} \vdash 2 * 2 = 4}}{\mathcal{T} \vdash \exists x(2 * x = 4)}$	$\dfrac{\dfrac{}{\vdash 4 = 4}}{\vdash \exists x(2 * x = 4)}$

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- General form (free variables are possible):

$$l \rightarrow r$$

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- and on **propositions** (predicate symbols):

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- we obtain a congruence modulo $\mathcal{R}$ (chosen set of rules): $\equiv$
- deduction rules transform as such:

$$\text{axiom}\ \frac{}{\Gamma, A \vdash A} \quad \text{becomes} \quad \frac{}{\Gamma, A \vdash B}\ \text{axiom},\ A \equiv B$$

Deduction modulo : natural deduction modulo - first presentation

$$\frac{}{\Gamma, A \vdash A}\text{axiom}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge\text{-i} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge\text{-e1} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge\text{-e2}$$

$$\Rightarrow\text{-i}\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \qquad \frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B} \Rightarrow\text{-e}$$

$$\frac{\Gamma \vdash \forall x A[x]}{\Gamma \vdash A[t]}\forall\text{-e, any } t \qquad \frac{\Gamma \vdash A[x]}{\Gamma \vdash \forall x A[x]}\forall\text{-i, } x \text{ free}$$

Deduction modulo : first presentation

Add then the following conversion rule:

$$\frac{\Gamma \vdash A}{\Gamma \vdash B}\; A \equiv B$$

Deduction modulo : natural deduction modulo, reloaded

$$\frac{}{\Gamma, A \vdash B}\text{axiom, } A \equiv B$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash C} \wedge\text{-i, } C \equiv A \wedge B \qquad \frac{\Gamma \vdash C}{\Gamma \vdash A} \wedge\text{-e1,} C \equiv A \wedge B \qquad \frac{\Gamma \vdash C}{\Gamma \vdash B} \wedge\text{-e2,} C \equiv A \wedge B$$

$$\Rightarrow\text{-i, } C \equiv A \wedge B \ \frac{\Gamma, A \vdash B}{\Gamma \vdash C} \qquad \frac{\Gamma \vdash C \qquad \Gamma \vdash A}{\Gamma \vdash B} \Rightarrow\text{-e, } C \equiv A \wedge B$$

$$\frac{\Gamma \vdash A[x]}{\Gamma \vdash B} \forall\text{-i, } x \text{ free,} B \equiv \forall x A[x] \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A[t]} \forall\text{-e, any } t, B \equiv \forall x A[x]$$

Example: 3

- consider the rewriting system $\mathcal{R}$:

$$\begin{array}{rcl} P(0) & \to & A \\ P(1) & \to & B \end{array}$$

$$\forall x P(x) \vdash A \wedge B$$

Example: 3

- consider the rewriting system $\mathcal{R}$:

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A Cut: a detour

$$\dfrac{\Gamma \vdash A \qquad \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\Rightarrow\text{-i}}{\Gamma \vdash B}\Rightarrow\text{-e}$$

- show $\Gamma \vdash A$
- show $\Gamma, A \vdash B$
- then, you have showed $\Gamma \vdash B$
- it is the application of a lemma.

A cut: a detour

$$\dfrac{\dfrac{\dfrac{\pi_1}{}}{\Gamma \vdash A} \quad \dfrac{\dfrac{\pi_2}{}}{\Gamma \vdash B}}{\dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge\text{-e}} \wedge\text{-i}$$

Replace it by π_1. And in the previous proof,

$$\dfrac{\dfrac{\theta}{\Gamma \vdash A} \quad \dfrac{\dfrac{\pi}{\Gamma, A \vdash B}}{\Gamma \vdash A \Rightarrow B} \Rightarrow\text{-i}}{\Gamma \vdash B} \Rightarrow\text{-e}$$

π is directly a proof of $\Gamma \vdash B$ *replace uses of A (nb: axioms) by* θ. In clear: don't use the lemma, reprove its instances.

General definition: a cut is an elimination plus an introduction (same symbol).

A cut: a detour

$$\dfrac{\dfrac{\theta}{\Gamma \vdash A'} \qquad \dfrac{\dfrac{\pi}{\Gamma, A \vdash B}}{\Gamma \vdash C} \Rightarrow\text{-i, } C \equiv A \Rightarrow B}{\Gamma \vdash B'} \Rightarrow\text{-e, } C \equiv A' \Rightarrow B'$$

- we show $\Gamma, A \vdash B$ and $\Gamma \vdash A$
- then we have showed $\Gamma \vdash B$.
- lemma: the good way for a human being.
- in practice: not adapted for automatic demonstration.

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- eliminating cuts: a central result.

$$\Gamma \vdash A \quad \rhd \quad \Gamma \vdash_{cf} A$$

- two main paths towards:
  - proof normalization (syntactic).
  - semantical methods.

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- two main paths towards:
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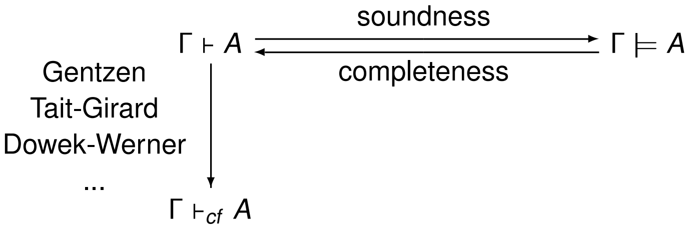
  - semantical methods.
- in deduction modulo: indecidable, need for conditions on $\mathcal{R}$.

The semantical method



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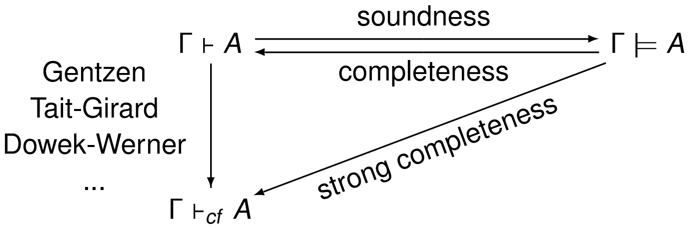

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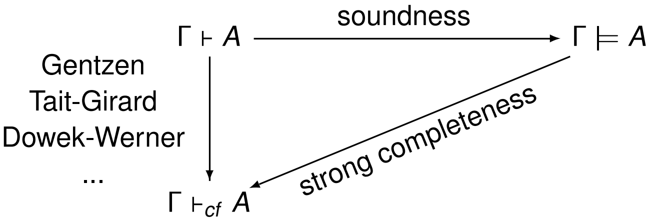

The normalization method(s)

- Curry-Howard: proofs = programs
- formulas = types
- proof tree = typing tree
- at the heart of proof assistants (PVS, Coq, Isabelle, ...)
- when a program calculates, it performs a cut elimination procedure.

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- formulas = types
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- at the heart of proof assistants (PVS, Coq, Isabelle, ...)
- when a program calculates, it performs a cut elimination procedure.
- show that all typables function terminates.

Curry-Howard correspondence

- Notation for proofs. Give a name to each of the hypothesis:

$$\Gamma = x_1 : A_1, \ldots, x_n : A_n$$

$$\frac{}{\Gamma, x : A \vdash x : A}\ \text{Axiom} \qquad \frac{\Gamma \vdash \pi : A \wedge B}{\Gamma \vdash \mathit{fst}(\pi) : A}\ \wedge\text{-e1}$$

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- **very** similar to a type system
- in deduction modulo, rewrite rules are silent:

$$\frac{\Gamma \vdash \pi : A}{\Gamma \vdash \pi : B}\ A \equiv B$$

Cut elimination with proof terms

- Cut elimination is a **process**, similar to function execution.

$$\dfrac{\dfrac{\Gamma \vdash \pi_1 : A \qquad \Gamma \vdash \pi_2 : B}{\Gamma \vdash \langle \pi_1, \pi_2 \rangle : A \wedge B}\,\wedge\text{-i}}{\Gamma \vdash fst(\langle \pi_1, \pi_2 \rangle) : A}\,\wedge\text{-e} \quad \rhd \quad \Gamma \vdash \pi_1 : A$$

$$\dfrac{\Gamma \vdash \theta : A \qquad \dfrac{\Gamma, x : A \vdash \pi : B}{\Gamma \vdash \lambda x.\pi : A \Rightarrow B}\,\Rightarrow\text{-i}}{\Gamma \vdash (\lambda x.\pi)\theta : B}\,\Rightarrow\text{-e} \quad \rhd \quad \Gamma \vdash \{\theta/x\}\pi : B$$

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- showing that every proof normalizes: the cut elimination process terminates.

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- to each formula A, associates a candidate $[\![A]\!]$. Show that if $\Gamma \vdash \pi : A$ then $\pi \in [\![A]\!]$.
- in deduction modulo, if $A \equiv B$, additional constraint:

$$[\![A]\!] = [\![B]\!]$$

Towards “usual” semantics

- such methods are defined in deduction modulo (Heyting arithmetic, higher-order logic, Zermelo’s set theory, ...)

Towards “usual” semantics

- such methods are defined in deduction modulo (Heyting arithmetic, higher-order logic, Zermelo’s set theory, ...)
- the sets of candidates have a structure: pseudo Heyting algebras [Dowek].

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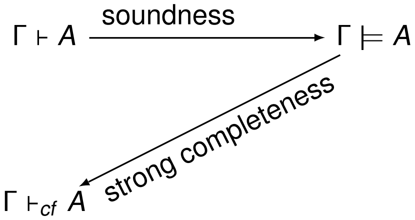
- an order
- operations on it: lowest upper bound (join: $\cup$ – pseudo union), greatest lower bound (meet: $\cap$ – intersection).

$$a \cap b \leq a \quad a \cap b \leq b \quad c \leq a \text{ and } c \leq b \text{ implies } c \leq a \cap b$$
$$a \leq a \cup b \quad b \leq a \cup b \quad a \leq c \text{ and } b \leq c \text{ implies } a \cup b \leq c$$

- think about $\mathbb{R}$ and closed sets (infinite l.u.b. is not infinite union)

Heyting algebras

Used in semantic cut elimination (Lipton, e.g.):



Heyting algebras

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- like boolean algebra, but with weaker complement.
- think about $\mathbb{R}$ and closed sets (infinite l.u.b. is not infinite union)

pseudo-Heyting algebras, aka Truth Values Algebras

- a universe Ω
- a pre-order: $a \leq b$ and $b \leq a$ with $a \neq b$ possible.
- operations on it: lowest upper bound (join: $\cup$ – pseudo union), greatest lower bound (meet: $\cap$ – intersection).

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Candidates form a pseudo-Heyting algebra

- $\top = \bot = \mathcal{SN}$
- $[\![A]\!] \cap [\![B]\!] = [\![A \wedge B]\!]$
- and so on.
- pre-order: trivial one.
- But $[\![A \wedge A]\!] \leq\geq [\![A]\!]$ only.

Super consistency

- consistency: there exists a model.

- condition in DM: $A \equiv B$ implies $[\![A]\!] = [\![B]\!]$

Super consistency

- consistency: there exists a model.
- super-consistency: for every (pseudo-Heyting) structure, there exists a model (interpretation).
- condition in DM: $A \equiv B$ implies $[\![A]\!] = [\![B]\!]$

Super consistency

- e.g. higher-order logic is super-consistent:

$$\begin{array}{rcl} M_\iota & = & \iota \text{ (dummy)} \\ M_o & = & \Omega \\ M_{t \to u} & = & M_u^{M_t} \end{array}$$

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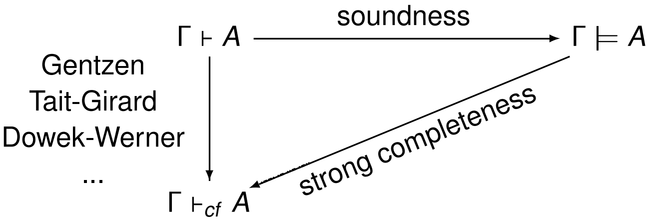
- hence, it has a model in the candidates pseudo-Heying Algebra
- $\Gamma \vdash \pi : A$ implies $\pi \in [\![A]\!]$.
- the system enjoys proof normalization.

Towards usual semantics



Towards usual semantics

- assuming we have a model $\mathcal{M}$, $[\![\_]\!]$ in the previous pseudo-Heyting algebra.
- first idea: pseudo-Heyting to Heyting by quotienting.

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- assuming we have a model $\mathcal{M}$, $[\![\_]\!]$ in the previous pseudo-Heyting algebra.
- first idea: pseudo-Heyting to Heyting by quotienting.
- trivial pseudo order implies $\top = \bot$.

The link: fibring

define

$$[A]^{\sigma}_{\phi} = [\![A]\!]_{\phi} \lhd A\sigma = \{\Gamma \mid \Gamma \vdash \pi : A\sigma, \pi \in [\![A]\!]_{\phi}\}$$

- $[\![A]\!]_{\phi}$ is a candidate of reducibility. It contains some proof terms $\Delta \vdash \nu : B$. We don't want them.

The link: fibring

define

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- $[\![A]\!]_{\phi}$ is a candidate of reducibility. It contains some proof terms $\Delta \vdash \nu : B$. We don't want them.
- **weak** definition: for *some* π only.
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- this proves semantical cut elimination.

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- proves directly $[A \wedge B] = [A] \cap [B]$. Usually, only:

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- proof resembles the proof for normalization.

Cut admissibility

Assume $\Gamma \vdash A$ has a proof (with cuts)

- $[\Gamma] \leq [A]$ in $\mathcal{D}$ by (usual) soundness
- $\Gamma \in [\Gamma]$
- $\Gamma \in [A]$ implies $\Gamma \vdash_{cf} A$
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- definition:

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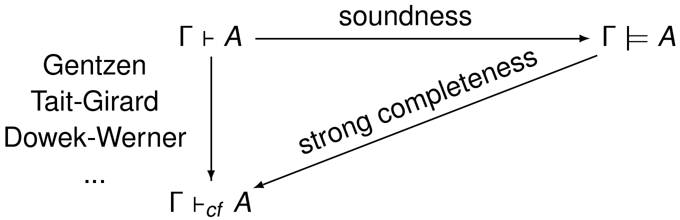
  is weak (some π only)
- we get (weak) normalization by evaluation.



- This diagram does not commute in deduction modulo.

Further work

- there is normalization by evaluation work, but in a Kripke style: links ?
- do the proof terms (candidates) always have a "pseudo-" structure ?
- realizing rewrite rule not with $\lambda x.x$ (not silently), could recover normalization and make the previous diagram commute again.

$$\frac{\Gamma \vdash \pi : A}{\Gamma \vdash \pi : B} \; A \equiv B$$