A simple proof that super consistency implies cut elimination

Olivier Hermant and Gilles Dowek

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Normalization:
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Tait-Girard
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...

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Normalization:
Gentzen
Tait-Girard
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- Normalization implies cut elimination but not the converse

Deduction Modulo (in a nutshell)

A framework integrating computation to deduction.

- A theory : a set of axioms and rewrite rules *e.g.*

$$x * 0 \longrightarrow 0$$

$$P(0) \longrightarrow \forall x Q(x)$$

defining a congruence $\equiv$

- Deduction rules (*e.g.* NJ) take $\equiv$ into account

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash A \Rightarrow B}{\Gamma \vdash B} \Rightarrow\text{-elim} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash C}{\Gamma \vdash C} \Rightarrow\text{-elim},\ C \equiv A \Rightarrow B$$

- Some theories have the cut elimination property, some do not

An example: simple-type theory (intuitionistic HOL)

- sorts: $\iota, o, \iota \longrightarrow o, \iota \longrightarrow \iota, \ldots$
-

$$\begin{array}{rcl} \alpha(\alpha(\alpha(S,x),y),z) & \longrightarrow & \alpha(\alpha(x,z),\alpha(y,z)) \\ \alpha(\alpha(K,x),y) & \longrightarrow & x \\ \varepsilon(\alpha(\alpha(\dot{\Rightarrow},x),y)) & \longrightarrow & \varepsilon(x) \Rightarrow \varepsilon(y) \\ \varepsilon(\alpha(\dot{\forall}_T,x)) & \longrightarrow & \forall y\ \varepsilon(\alpha(x,y)) \end{array}$$

- first-order encoding of simple-type theory + orientation

Truth values algebras

- Heyting algebra:
  - an ordered set with g.l.b. (to interprete $\wedge$, $\forall$ and $\top$) and l.u.b. (to interprete $\vee$, $\exists$ and $\bot$) and $\rightarrow$ (to interprete $\Rightarrow$)
  - like boolean algebra but with weaker complement
- Truth value algebra : same as Heyting algebra but order replaced by pre-order

Truth values algebra based models

- Propositions interpreted in a TVA $\mathcal{B}$
- we keep soundness and completeness
- in deduction modulo, additional constraint:

$$A \equiv B \text{ implies } [\![A]\!] = [\![B]\!]$$

- Notice

$$A \Leftrightarrow B \text{ (only) implies } ([\![A]\!] \leq [\![B]\!] \text{ and } [\![B]\!] \leq [\![A]\!])$$

Super consistency

$\equiv$ is super-consistent if for all TVA $\mathcal{B}$ it has a $\mathcal{B}$-valued model

- reducibility candidates form a TVA (and not a HA!)
- super-consitency implies normalization (Dowek)
- hence super-consitency implies cut elimination
- we give here a simpler proof

the Algebra of sequents $\mathcal{S}$

- we simplify the algebra "candidates of reducibility"
- reducibility candidates are sets of proofs
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- the sequents truth value $\tilde{\bot}$: set of sequents that have a neutral cut-free proof
- super-consistency implies the existence of a model $\mathcal{M}$ where $[\![A]\!]_\phi$ is an element of $\mathcal{S}$

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- $\mathcal{S}$ is not a Heyting Algebra. Can we get back one ?
- turning the pre-order into an order (quotienting) would give a trivial HA ($\top = \bot$).
- instead, we define some fibration over A:

$$[A] = \{\Gamma \mid (\Gamma \vdash A) \in [\![A]\!]\}$$

$$[A]^{\sigma}_{\phi} = \{\Gamma \mid (\Gamma \vdash \sigma A) \in [\![A]\!]_{\phi}\}$$

Some facts about $\mathcal{S}$ and Ω

$\mathcal{S}$	Ω	
$(\Gamma, A \vdash A) \in a$	$\Gamma, A \in [A]$	axiom
$(\Gamma \vdash A) \in a$ iff $(\Gamma \vdash B) \in a$	$[A] = [B]$	if $B \equiv A$
$(\Gamma \vdash A) \in b$	$\Gamma \in [A]$	$\Rightarrow \Gamma \vdash_{cf} A$

Key lemma: [ ] defines almost a model interpretation !

- $[\bot]$ is the least element of Ω.
- $[A \wedge B] = [A] \cap [B]$
- $[\forall x A] = \bigcap [A]^{t/x}_{d/x}$ with $d \in M$, t closed term.

Only missing to get a model: the domain!

- hybridization $D = \mathcal{T} \times M = \{\langle t, d\rangle\}$.
- interpretation for symbols

$$\begin{array}{rcl}
\hat{f}^{\mathcal{D}}(\langle t_1, d_1\rangle, \ldots, \langle t_n, d_n\rangle) & = & \langle (f(t_1, \ldots, t_n), \hat{f}^{\mathcal{M}}(d_1, \ldots, d_n)\rangle \\
\hat{P}^{\mathcal{D}}(\langle t_1, d_1\rangle, \ldots, \langle t_n, d_n\rangle) & = & [(t_1/x_1, \ldots, t_n/x_n)P]_{(d_1/x_1, \ldots, d_n/x_n)} \\
 & = & \{\Gamma \mid (\Gamma \vdash P(t_1, \ldots, t_n)) \in [\![P]\!]_{(d_1/x_1, \ldots, d_n/x_n)}\}
\end{array}$$

- remember: M given by super-consistency applied to $\mathcal{S}$.
  Embedding a (possibly) complex structure at the term level.

Finally the theorem ...

Assume $\Gamma \vdash A$ has a proof (with cuts)

- $\Gamma \in [\wedge\Gamma]$
- $[\Gamma] \leq [A]$ in $\mathcal{D}$ by (usual) soundness
- $\Gamma \in [A]$ implies $\Gamma \vdash_{cf} A$
- Q.E.D.

A case study: HOL

Super-consistency constructs the following $\mathcal{M}$-valued Algebra:

- the domain is, respectively for each type
  - $M_o = \mathcal{S}$ (anticipating that o is the type of "propositional content")
  - $M_\iota = \{0\}$ (or any other "dummy" constant)
  - $M_{T \to U} = M_U^{M_T}$ (functional space)
- the (immediate) interpretation for the symbols:
  - $\hat{\varepsilon} : a \mapsto a$.
  - $\hat{\dot{\wedge}} = \tilde{\wedge}$ (the operation of $\mathcal{M}$), ...

Exporting this into $\mathcal{D}$ (the Ω-valued model):

- $D_o = \{\langle t, d\rangle\}$, t closed term of sort o, $d \in \mathcal{S}$.
- $D_\iota = \{\langle t, 0\rangle\}$ (dummy constant)
- $D_{T\to U} = \{\langle t, f\rangle\}$ with t of sort $T \to U$ and $f \in M_U^{M_T}$.
- application is pointwise: $\hat{\alpha}(\langle t, f\rangle, \langle u, g\rangle) = \langle tu, f(g)\rangle$.
- re-inventing (and simplifying) V-complexes

The V-complexes semantic method

Takahashi, Prawitz, Andrews, Okada, De Marco, Lipton ...
Idea: find a (Heyting-valued) model such that $[\![\Gamma]\!] \leq [\![A]\!]$ implies $\Gamma \vdash_{cf} A$. Take care of intensionality and impredicativity !

- the Heyting Algebra has for basis $[A] = \{\Gamma \mid \Gamma \vdash_{cf} A\}$
- The construction of the domains D has to be intricated. Requires accuracy.
- Our construction in two steps (thanks to the choice $M_o = S$ and not $\{0, 1\}$) avoids this.

Comparison

We have two semantical methods:

	V-complexes	Hybridization
applies to	HOL	any case (including HOL)
D_o	$\subset \mathcal{T} \times \Omega$	$= \mathcal{T} \times \mathcal{S}$
$\langle t, f \rangle \bullet \langle u, g \rangle$	$f(\langle u, g \rangle)$	$\langle tu, f(g) \rangle$

This all comes from $\Omega \neq \mathcal{S}$.

- future work: extension to normalization ? extension to non super-consistent theories ?
- Heyting (v.s. Kripke) fight back (NBE : Coquand, Altenkirch, Hofman, Streicher)
- Reverse engineering ? i.e. Could this helps understand the historical V-complexes ? Generalize them ?