Semantic Cut Elimination in Sequent Calculus

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- Is there a link with normalisation method ?

The Intuitionnistic Sequent Calculus

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- $\Vdash$ is a relation between worlds and propositions, verifying certain condition, among which:
  - if $\alpha \leq \beta$ and $\alpha \Vdash P$, then $\beta \Vdash P$
  - $\alpha \Vdash P \Rightarrow Q$ iff for any $\beta \geq \alpha$ we have $\beta \Vdash P$ implies $\beta \Vdash Q$.

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- We can no more use usual definitions (e.g. completeness or saturation), because we can't no more ensure that $\Gamma \vdash^{cf} P$ implies $P \in \Gamma$.
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- Adding the fact that we are in an intuitionnistic framework, they become:
  - Consistency: $\Gamma \nvdash^{cf} A$
  - Completeness (saturation): $\Gamma, P \vdash^{cf} A$ or $P \in \Gamma$
  - Henkin witnesses: $\Gamma, \exists x P \nvdash^{cf} A$ implies $\{c/x\}P \in \Gamma$ for some constant c.

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- This **is** a Kripke Structure. Given a ($\bot$-)consistent theory T, we complete it into Γ, and in the KS previously defined, we have $\Gamma \Vdash T$. $\square$

The Deduction Modulo

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- Rewrite rules on proposition is the central paradigm of deduction modulo.

The Deduction Modulo

Examples ?

The Intuitionnistic Sequent Calculus Modulo

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- The previous method should then be modified, because we have to ensure this new condition.

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- The model definition depends on the class of the rewrite rules. (since we don't have cut-elimination for all the rewrite system)

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- Confluence is necessary, in order not to have:

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ B \vee C & & B' \wedge C' \end{tikzcd}$$

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- We extend it by induction on the order $\prec$ to *non atomic and non-normal* propositions:
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- From the well-foundedness of $\prec$, the above definition is well-founded.

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- As a corollary, we get the cut-elimination theorem:
  Theorem: Let $\mathcal{R}$ be a confluent Rewrite System compatible with a wfo having the subformula property.

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A counterexample to normalization

The rewrite system composed of the following rewrite rule:

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- Has the cut elimination property. We can find by other means a cut-free proof of $\vdash^{cf}_{\mathcal{R}} A \Rightarrow A$

A counterexample to normalization

Theorem:

Further work

- Can we extend these results
- Links with methods based on Heyting Algebras ?