A simple proof that super-consistency implies cut elimination

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Abstract. We give a simple proof of the cut elimination theorem for super-consistent theories in natural deduction modulo, inspired by proof normalization, but without explicit proofs. It can also be formulated as a completeness proof for the cut free calculus. This formulation involves some kind of V-complexes. We then discuss application to simple types theory and the links, in this case, with the normalization and purely semantical methods, in particular those using V-complexes.

1 Introduction

Deduction modulo is an extension of predicate logic where some axioms may be replaced by rewrite rules. For instance, the axiom $x + 0 = x$ may be replaced by the rewrite rule $x + 0 \longrightarrow x$ and the axiom $x \subseteq y \Leftrightarrow \forall z\ (z \in x \Rightarrow z \in y)$ may be replaced by the rewrite rule $x \subseteq y \longrightarrow \forall z\ (z \in x \Rightarrow z \in y)$.

In the model theory of deduction modulo, it is important to distinguish the fact that some propositions are computationally equivalent (*e.g.* $x \subseteq y$ and $\forall z\ (z \in x \Rightarrow z \in y)$) in which case, they should have the same value in a model, from the fact that they are provably equivalent (*e.g.* Pythagora's Theorem and Thales' Theorem) in which case they may have different values. This has lead, in [3], to introduce a generalization of Heyting algebras called *truth-value algebras* and a notion of $\mathcal{B}$-valued model, where $\mathcal{B}$ is a truth value algebra. We have called *super-consistent* the theories that have a $\mathcal{B}$-valued model for all truth values algebras $\mathcal{B}$ and we have given examples of consistent theories that are not super-consistent.

In deduction modulo, in some theories, some proofs do not normalize. For instance, in the theory formed with the rewrite rule $P \longrightarrow (P \Rightarrow Q)$, the proposition Q has a proof but no cut free proof. In some other theories, such as the theory formed with the rewrite rule $P \longrightarrow (Q \Rightarrow P)$, all proofs strongly normalize. We have proved in [3] that all proofs normalize in all super-consistent theories. This proof proceeds by observing that reducibility candidates [6] can be structured in a truth value algebra and thus that super-consistent theories have reducibility candidate valued models. Then, the existence of such a model implies proof normalization and hence cut elimination [5]. As many theories, in particular arithmetic and simple type theory, are super-consistent, we get back Gentzen's and Girard's theorem as corollaries.

This paper is an attempt to simplify this proof replacing the algebra of reducibility candidates by a simpler truth value algebra. Reducibility candidates are sets of proofs and we show that we can replace each proof of such a set by its conclusion, obtaining this way sets of sequents, rather than sets of proofs, for truth values. The proof we obtain this way is not a strong normalization proof but just a cut elimination proof.

Although the truth values of our model are sets of sequents, our cut elimination proof makes use of another truth value algebra, that happens to be a Heyting algebra: the algebra of contexts.

We can build another model where truth values are contexts, but this requires to enlarge the domain of the model using a technique of *hybridization*. The elements of such an hybrid model are quite similar to the V-complexes used in the semantic proofs of cut elimination for simple type theory [10, 11, 1, 2, 7]. Thus, we show that these proofs can be simplified using an alternative notion of V-complex and also that the V-complexes introduced for proving cut elimination of simple type theory can be used for other theories as well. Finally, this hybridization technique allows to have a uniform view of the two main techniques used to prove cut elimination: normalization and model theoretic completeness of the cut free calculus.

2 Super-consistency

To keep the paper self contained, we recall in this section the definition of deduction modulo, truth values algebras, $\mathcal{B}$-valued models and super-consistency. A more detailed presentation can be found in [3].

2.1 Deduction modulo

Deduction modulo [4, 5] is an extension of predicate logic (either single-sorted or many-sorted predicate logic) where a theory is defined by a set of axioms $\mathcal{T}$ and a congruence $\equiv$ defined by a confluent rewrite system rewriting terms to terms and atomic propositions to propositions. The deduction rules, for instance the natural deduction rules, are modified to take the congruence $\equiv$ into account. For example, the *modus ponens* rule is not stated as usual

$$\frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$$

but

$$\frac{\Gamma \vdash C \quad \Gamma \vdash A}{\Gamma \vdash B}\; C \equiv A \Rightarrow B$$

In deduction modulo, some proofs, in some theories do not normalize. For instance, in the theory formed with the rewrite rule $P \longrightarrow (P \Rightarrow Q)$, the proposition Q has a proof but no cut free proof. In some other theories, such as the theory formed with the rewrite rule $P \longrightarrow (Q \Rightarrow P)$, all proofs strongly normalize.

In deduction modulo, like in predicate logic, normal proofs of a sequent of the form $\vdash A$ always end with an introduction rule. Thus, when a theory can be expressed in deduction modulo with rewrite rules only, *i.e.* with no axioms, in such a way that proofs modulo these rewrite rules strongly normalize, then the theory is consistent, it has the disjunction property and the witness property, various proof search methods for this theory are complete, ...

Many theories can be expressed this way in deduction modulo, in particular arithmetic and simple type theory and the notion of cut of deduction modulo subsumes the *ad hoc* notions of cut defined for these theories.

For instance, simple type theory can be defined as follows.

Definition 1 (Simple type theory). *The sorts are inductively defined by ι and o are sorts and if T and U are sorts then $T \to U$ is a sort. The language contains the constants $S_{T,U,V}$ of sort $(T \to U \to V) \to (T \to U) \to T \to V$, $K_{T,U}$ of sort $T \to U \to T$, $\dot{\top}$ of sort o and $\dot{\bot}$ of sort o, $\dot{\Rightarrow}$, $\dot{\wedge}$ and $\dot{\vee}$ of sort $o \to o \to o$, $\dot{\forall}_T$ and $\dot{\exists}_T$ of sort $(T \to o) \to o$, the function symbols $\alpha_{T,U}$ of rank $\langle T \to U, T, U\rangle$ and the predicate symbol ε of rank $\langle o \rangle$.*

The rules are

$$\alpha(\alpha(\alpha(S_{T,U,V}, x), y), z) \longrightarrow \alpha(\alpha(x, z), \alpha(y, z))$$

$$\alpha(\alpha(K_{T,U}, x), y) \longrightarrow x$$

$$\varepsilon(\dot{\top}) \longrightarrow \top$$

$$\varepsilon(\dot{\bot}) \longrightarrow \bot$$

$$\varepsilon(\alpha(\alpha(\dot{\Rightarrow}, x), y)) \longrightarrow \varepsilon(x) \Rightarrow \varepsilon(y)$$

$$\varepsilon(\alpha(\alpha(\dot{\wedge}, x), y)) \longrightarrow \varepsilon(x) \wedge \varepsilon(y)$$

$$\varepsilon(\alpha(\alpha(\dot{\vee}, x), y)) \longrightarrow \varepsilon(x) \vee \varepsilon(y)$$

$$\varepsilon(\alpha(\dot{\forall}, x)) \longrightarrow \forall y\ \varepsilon(\alpha(x, y))$$

$$\varepsilon(\alpha(\dot{\exists}, x)) \longrightarrow \exists y\ \varepsilon(\alpha(x, y))$$

2.2 Truth values algebras

Definition 2 (Truth values algebra). *Let $\mathcal{B}$ be a set, whose elements are called* truth values, *$\mathcal{B}^+$ be a subset of $\mathcal{B}$, whose elements are called* positive *truth values, $\mathcal{A}$ and $\mathcal{E}$ be subsets of $\wp(\mathcal{B})$, $\tilde{\top}$ and $\tilde{\bot}$ be elements of $\mathcal{B}$, $\tilde{\Rightarrow}$, $\tilde{\wedge}$, and $\tilde{\vee}$ be functions from $\mathcal{B} \times \mathcal{B}$ to $\mathcal{B}$, $\tilde{\forall}$ be a function from $\mathcal{A}$ to $\mathcal{B}$ and $\tilde{\exists}$ be a function from $\mathcal{E}$ to $\mathcal{B}$. The structure $\mathcal{B} = \langle \mathcal{B}, \mathcal{B}^+, \mathcal{A}, \mathcal{E}, \tilde{\top}, \tilde{\bot}, \tilde{\Rightarrow}, \tilde{\wedge}, \tilde{\vee}, \tilde{\forall}, \tilde{\exists}\rangle$ is said to be* a truth value algebra *if the set $\mathcal{B}^+$ is* closed by the intuitionistic deduction rules i.e. *if for all a, b, c in $\mathcal{B}$, A in $\mathcal{A}$ and E in $\mathcal{E}$,*

1. *if* $a \tilde{\Rightarrow} b \in \mathcal{B}^+$ *and* $a \in \mathcal{B}^+$ *then* $b \in \mathcal{B}^+$,
2. $a \tilde{\Rightarrow} b \tilde{\Rightarrow} a \in \mathcal{B}^+$,
3. $(a \tilde{\Rightarrow} b \tilde{\Rightarrow} c) \tilde{\Rightarrow} (a \tilde{\Rightarrow} b) \tilde{\Rightarrow} a \tilde{\Rightarrow} c \in \mathcal{B}^+$,
4. $\tilde{\top} \in \mathcal{B}^+$,
5. $\tilde{\bot} \tilde{\Rightarrow} a \in \mathcal{B}^+$,
6. $a \tilde{\Rightarrow} b \tilde{\Rightarrow} (a \tilde{\wedge} b) \in \mathcal{B}^+$,
7. $(a \tilde{\wedge} b) \tilde{\Rightarrow} a \in \mathcal{B}^+$,
8. $(a \tilde{\wedge} b) \tilde{\Rightarrow} b \in \mathcal{B}^+$,
9. $a \tilde{\Rightarrow} (a \tilde{\vee} b) \in \mathcal{B}^+$,
10. $b \tilde{\Rightarrow} (a \tilde{\vee} b) \in \mathcal{B}^+$,
11. $(a \tilde{\vee} b) \tilde{\Rightarrow} (a \tilde{\Rightarrow} c) \tilde{\Rightarrow} (b \tilde{\Rightarrow} c) \tilde{\Rightarrow} c \in \mathcal{B}^+$,
12. *the set* $a \tilde{\Rightarrow} A = \{a \tilde{\Rightarrow} e \mid e \in A\}$ *is in* $\mathcal{A}$ *and the set* $E \tilde{\Rightarrow} a = \{e \tilde{\Rightarrow} a \mid e \in E\}$ *is in* $\mathcal{A}$,
13. *if all elements of* A *are in* $\mathcal{B}^+$ *then* $\tilde{\forall}\ A \in \mathcal{B}^+$,
14. $\tilde{\forall}\ (a \tilde{\Rightarrow} A) \tilde{\Rightarrow} a \tilde{\Rightarrow} (\tilde{\forall}\ A) \in \mathcal{B}^+$,
15. *if* $a \in A$, *then* $(\tilde{\forall}\ A) \tilde{\Rightarrow} a \in \mathcal{B}^+$,
16. *if* $a \in E$, *then* $a \tilde{\Rightarrow} (\tilde{\exists}\ E) \in \mathcal{B}^+$,
17. $(\tilde{\exists}\ E) \tilde{\Rightarrow} \tilde{\forall}\ (E \tilde{\Rightarrow} a) \tilde{\Rightarrow} a \in \mathcal{B}^+$.

Remark. Any Heyting algebra is a truth value algebra. The operations $\tilde{\top}$, $\tilde{\wedge}$, $\tilde{\forall}$ are greatest lower bounds, the operations $\tilde{\bot}$, $\tilde{\vee}$, $\tilde{\exists}$ least upper bounds, the operation $\tilde{\Rightarrow}$ the arrow of the Heyting algebra and $\mathcal{B}^+ = \{\tilde{\top}\}$.

Definition 3 (Full). *A truth values algebra is said to be* full *if* $\mathcal{A} = \mathcal{E} = \wp(\mathcal{B})$, i.e. *if* $\tilde{\forall}\ A$ *and* $\tilde{\exists}\ A$ *exist for all subsets* A *of* $\mathcal{B}$.

Definition 4 (Ordered truth values algebra). *An* ordered truth values algebra *is a truth values algebra together with a relation* $\sqsubseteq$ *on* $\mathcal{B}$ *such that*

- $\sqsubseteq$ *is an order relation,*
- $\mathcal{B}^+$ *is upward closed,*
- $\tilde{\top}$ *and* $\tilde{\bot}$ *are maximal and minimal elements.*
- $\tilde{\wedge}$, $\tilde{\vee}$, $\tilde{\forall}$ *and* $\tilde{\exists}$ *are monotonous,* $\tilde{\Rightarrow}$ *is left anti-monotonous and right monotonous.*

Definition 5 (Complete ordered truth values algebra). *A ordered truth values algebra is said to be* complete *if every subset of* $\mathcal{B}$ *has a greatest lower bound for* $\sqsubseteq$.

2.3 Models

Definition 6 ($\mathcal{B}$-structure). *Let* $\mathcal{L} = \langle f_i, P_j \rangle$ *be a language in predicate logic and* $\mathcal{B}$ *be a truth values algebra, a* $\mathcal{B}$-structure *for the language* $\mathcal{L}$, $\mathcal{M} = \langle \mathcal{M}, \mathcal{B}, \hat{f}_i, \hat{P}_j \rangle$ *is a structure such that* $\hat{f}_i$ *is a function from* $\mathcal{M}^n$ *to* $\mathcal{M}$ *where* n *is the arity of the symbol* f_i *and* $\hat{P}_j$ *is a function from* $\mathcal{M}^n$ *to* $\mathcal{B}$ *where* n *is the arity of the symbol* P_i.

This definition extends trivially to many-sorted languages.

Definition 7 (Denotation). *Let $\mathcal{B}$ be a truth values algebra, $\mathcal{M}$ be a $\mathcal{B}$-structure and ϕ be an assignment. The denotation $[\![A]\!]_\phi$ of a proposition A in $\mathcal{M}$ is defined as follows*

- $[\![x]\!]_\phi = \phi(x)$,
- $[\![f(t_1, ..., t_n)]\!]_\phi = \hat{f}([\![t_1]\!]_\phi, ..., [\![t_n]\!]_\phi)$,
- $[\![P(t_1, ..., t_n)]\!]_\phi = \hat{P}([\![t_1]\!]_\phi, ..., [\![t_n]\!]_\phi)$,
- $[\![\top]\!]_\phi = \tilde{\top}$,
- $[\![\bot]\!]_\phi = \tilde{\bot}$,
- $[\![A \Rightarrow B]\!]_\phi = [\![A]\!]_\phi \mathrel{\tilde{\Rightarrow}} [\![B]\!]_\phi$,
- $[\![A \wedge B]\!]_\phi = [\![A]\!]_\phi \mathrel{\tilde{\wedge}} [\![B]\!]_\phi$,
- $[\![A \vee B]\!]_\phi = [\![A]\!]_\phi \mathrel{\tilde{\vee}} [\![B]\!]_\phi$,
- $[\![\forall x\ A]\!]_\phi = \tilde{\forall}\ \{[\![A]\!]_{\phi+(e/x)} \mid e \in \mathcal{M}\}$,
- $[\![\exists x\ A]\!]_\phi = \tilde{\exists}\ \{[\![A]\!]_{\phi+(e/x)} \mid e \in \mathcal{M}\}$.

Notice that the denotation of a proposition containing quantifiers may be undefined, but it is always defined if the truth value algebra is full.

Definition 8 (Model). *A proposition A is said to be* valid *in a $\mathcal{B}$-structure $\mathcal{M}$, and the $\mathcal{B}$-structure $\mathcal{M}$ is said to be* model of A, *$\mathcal{M} \models A$, if for all assignments ϕ, $[\![A]\!]_\phi$ is defined and is a positive truth value.*

Let $\mathcal{T}, \equiv$ be a theory in deduction modulo. The $\mathcal{B}$-structure $\mathcal{M}$ is said to be a model of the theory $\mathcal{T}, \equiv$ if all axioms of $\mathcal{T}$ are valid in $\mathcal{M}$ and for all terms or propositions A and B such that $A \equiv B$ and assignment ϕ, $[\![A]\!]_\phi$ and $[\![B]\!]_\phi$ are defined and $[\![A]\!]_\phi = [\![B]\!]_\phi$.

Proposition 1 (Soundness and completeness). *$\mathcal{T} \vdash_\equiv A$ if and only if A is valid in all the models of $\mathcal{T}, \equiv$ where the truth values algebra is full, ordered and complete.*

2.4 Super-consistency

Definition 9 (Super-consistent). *A theory $\mathcal{T}, \equiv$ in deduction modulo is* super-consistent *if it has a $\mathcal{B}$-valued model for all full, ordered and complete truth values algebras $\mathcal{B}$.*

Proposition 2. *Simple type theory is super-consistent.*

Proof. Let $\mathcal{B}$ be a full truth values algebra. The model $\mathcal{M}_\iota = \{0\}$, $\mathcal{M}_o = \mathcal{B}$, $\mathcal{M}_{T \to U} = \mathcal{M}_U^{\mathcal{M}_T}$, $\hat{S}_{T,U,V} = a \mapsto (b \mapsto (c \mapsto a(c)(b(c))))$, $\hat{K}_{T,U} = a \mapsto (b \mapsto a)$, $\hat{\alpha}(a, b) = a(b)$, $\hat{\varepsilon}(a) = a$, $\hat{\dot{\top}} = \tilde{\top}$, $\hat{\dot{\bot}} = \tilde{\bot}$, $\hat{\dot{\Rightarrow}} = \tilde{\Rightarrow}$, $\hat{\dot{\wedge}} = \tilde{\wedge}$, $\hat{\dot{\vee}} = \tilde{\vee}$, $\hat{\dot{\forall}}_T = a \mapsto \tilde{\forall}_T(Range(a))$, $\hat{\dot{\exists}}_T = a \mapsto \tilde{\exists}_T(Range(a))$ where $Range(a)$ is the range of the function a, is a $\mathcal{B}$-valued model of simple type theory.

3 Cut elimination

3.1 The algebra of sequents

We are now ready to prove that super-consistent theories have the cut elimination property.

Definition 10 (Neutral proof). *A proof is said to be* neutral *if its last rule is the axiom rule or an elimination rule, but not an introduction rule.*

Definition 11 (A positive definition of cut free proofs). Cut free *proofs are defined inductively as follows:*

- *a proof that ends with the axiom rule is cut free,*
- *a proof that ends with an introduction rule and where the premises of the last rule are proved with cut free proofs is cut free,*
- *a proof that ends with an elimination rule and where the major premise of the last rule is proved with a neutral cut free proof and the other premises with cut free proofs is cut free.*

Definition 12 (The algebra of sequents).

- $\tilde{\top}$ *is the set of sequents* $\Gamma \vdash C$ *that have a neutral cut free proof or such that* $C \equiv \top$.
- $\tilde{\bot}$ *is the set of sequents* $\Gamma \vdash C$ *that have a neutral cut free proof.*
- $a\ \tilde{\wedge}\ b$ *is the set of sequents* $\Gamma \vdash C$ *that have a neutral cut free proof or such that* $C \equiv (A \wedge B)$ *with* $(\Gamma \vdash A) \in a$ *and* $(\Gamma \vdash B) \in b$.
- $a\ \tilde{\vee}\ b$ *is the set of sequents* $\Gamma \vdash C$ *that have a neutral cut free proof or such that* $C \equiv (A \vee B)$ *with* $(\Gamma \vdash A) \in a$ *or* $(\Gamma \vdash B) \in b$.
- $a \mathrel{\tilde{\Rightarrow}} b$ *is the set of sequents* $\Gamma \vdash C$ *that have a neutral cut free proof or such that* $C \equiv (A \Rightarrow B)$ *and for all contexts* Σ *such that* $(\Gamma, \Sigma \vdash A) \in a$, *we have* $(\Gamma, \Sigma \vdash B) \in b$.
- $\tilde{\forall}\ S$ *is the set of sequents* $\Gamma \vdash C$ *that have a neutral cut free proof or such that* $C \equiv (\forall x\ A)$ *and for every term* t *and every* a *in* S, $(\Gamma \vdash (t/x)A) \in a$.
- $\tilde{\exists}\ S$ *is the set of sequents* $\Gamma \vdash C$ *that have a neutral cut free proof or such that* $C \equiv (\exists x\ A)$ *and for some term* t *and some* a *in* S, $(\Gamma \vdash (t/x)B) \in a$.

Let $\mathcal{S}$ *is the smallest set of sets of sequents closed by* $\tilde{\top}$, $\tilde{\bot}$, $\tilde{\wedge}$, $\tilde{\vee}$, $\tilde{\Rightarrow}$, $\tilde{\forall}$, $\tilde{\exists}$ *and by by arbitrary intersections.*

The structure $\langle \mathcal{S}, \mathcal{S}, \wp(\mathcal{S}), \wp(\mathcal{S}), \tilde{\top}, \tilde{\bot}, \tilde{\Rightarrow}, \tilde{\wedge}, \tilde{\vee}, \tilde{\forall}, \tilde{\exists}, \subseteq \rangle$ *is a full, ordered and complete truth values algebra.*

Proposition 3. *Let* a *be an element of* $\mathcal{S}$.

- $(\Gamma, A \vdash A) \in a$.
- *If* $(\Gamma \vdash B) \in a$ *then* $(\Gamma, A \vdash B) \in a$.
- *If* $(\Gamma \vdash A) \in a$ *and* $B \equiv A$ *then* $(\Gamma \vdash B) \in a$.
- *If* $(\Gamma \vdash A) \in a$ *then* $\Gamma \vdash A$ *has a cut free proof.*

Proof. The first proposition is proved by noticing that the sequent $\Gamma, A \vdash A$ has a neutral cut free proof. The others are simple inductions on the construction of a.

Remark. The algebra $\mathcal{S}$ is not a Heyting algebra. In particular $\tilde{\top} \mathrel{\tilde{\wedge}} \tilde{\top}$ is different from $\tilde{\top}$: the first set contains the sequent $\vdash \top \wedge \top$, but not the second.

Consider a super-consistent theory defined by a confluent rewrite system. Let $\equiv$ be the congruence generated by this rewrite system. Let $\mathcal{T}$ be the set of open terms in the language of this theory.

As this theory is super-consistent, it has a $\mathcal{S}$-model. Let $\mathcal{M}$ be this model. As usual, the domain of $\mathcal{M}$ will also be called $\mathcal{M}$.

3.2 The algebra of contexts

Definition 13 (Projection). *Let b be a set of sequents and A a formula, we define the projection of b along A, $b \lhd A$ as the set of contexts Γ such that $(\Gamma \vdash A) \in b$.*

Definition 14 (Outer value [8, 9]). *Let A be a proposition and ϕ be a valuation and σ a substitution, we define the set of contexts $[A]^\sigma_\phi$ as the set $[\![A]\!]_\phi \lhd \sigma A$* i.e. $\{\Gamma \mid (\Gamma \vdash \sigma A) \in [\![A]\!]_\phi\}$.

Proposition 4.
- $(\Gamma, \sigma A) \in [A]^\sigma_\phi$.
- *If $\Gamma \in [B]^\sigma_\phi$ then $\Gamma, A \in [B]^\sigma_\phi$.*
- *If $\Gamma \in [A]^\sigma_\phi$ and $B \equiv A$ then $\Gamma \in [B]^\sigma_\phi$.*
- *If $\Gamma \in [A]^\sigma_\phi$ then $\Gamma \vdash \sigma A$ has a cut free proof.*

Proof. From Proposition 3.

Definition 15 (The algebra of contexts). *The set Ω is the smallest set of set of contexts containing all the $[A]_\psi$ for some proposition A and assignment ψ and closed by arbitrary intersections.*

Notice that each element c of Ω can be written of the form

$$c = \bigcap [A_i]^{\sigma_i}_{\phi_i}$$

Proposition 5. *The set Ω ordered by inclusion is a complete Heyting algebra.*

Proof. The maximum element $\check{\top}$ is $[\top]$, it is the set of all contexts, as, by Definition 12, for any context Γ, $(\Gamma \vdash \top) \in \tilde{\top}$. The minimum element $\check{\bot}$ is the intersection of all elements of Ω. The binary infimum $\check{\wedge}$ is binary intersection. The infinitary infimum $\check{\forall}$ is infinitary intersection. The binary supremum of a and b $a \mathrel{\check{\vee}} b$ is the intersection of all the elements that contain $a \cup b$

$$a \mathrel{\check{\vee}} b = \bigcap_{(a \cup b) \subseteq c} c$$

Notice that, in this definition, we could also restrict to the case where c has the form $[C]^\tau_\rho$. The infinitary supremum $\check{\exists}\ S$ of a the elements of a set S is the intersection of all the elements that contain the union of the elements of S

$$\check{\exists}\ S = \bigcap_{(\bigcup S) \subseteq c} c$$

In this definition, we could again also restrict to the case where c has the form $[C]^\tau_\rho$. Finally, the arrow $\check{\Rightarrow}$ of two elements a and b is the supremum of all the c in Ω such that $a \cap c \leq b$

$$a \check{\Rightarrow} b = \check{\exists}\ \{c \in \Omega \mid a \cap c \leq b\}$$

Notice that Ω is a non trivial Heyting algebra, although the truth values algebra $\mathcal{S}$ is trivial in the sense that all truth values are positive and that is ins not the quotient $\mathcal{S}/\mathcal{S}^+$ that is a trivial Heyting algebra.

Proposition 6 (Key lemma).

- $[\top]^\sigma_\phi = \check{\top}$,
- $[\bot]^\sigma_\phi = \check{\bot}$,
- $[A \wedge B]^\sigma_\phi = [A]^\sigma_\phi\ \check{\wedge}\ [B]^\sigma_\phi$,
- $[A \vee B]^\sigma_\phi = [A]^\sigma_\phi\ \check{\vee}\ [B]^\sigma_\phi$,
- $[A \Rightarrow B]^\sigma_\phi = [A]^\sigma_\phi \check{\Rightarrow} [B]^\sigma_\phi$,
- $[\forall x A]^\sigma_\phi = \check{\forall}\ \{[A]^{\sigma+(t/x)}_{\phi+(d/x)} \mid t \in \mathcal{T}, d \in \mathcal{M}\}$,
- $[\exists x A]^\sigma_\phi = \check{\exists}\ \{[A]^{\sigma+(t/x)}_{\phi+(d/x)} \mid t \in \mathcal{T}, d \in \mathcal{M}\}$.

Proof. – By definition of $\check{\top}$.

- The set $\check{\bot}$ is the intersection of all $[C]^\tau_\rho$. In particular, $\check{\bot} \subseteq [\bot]^\sigma_\phi$. Conversely, let $\Gamma \in [\bot]^\sigma_\phi$. Consider arbitrary C, ρ and τ. By Definition 12, $\Gamma \vdash \sigma\bot$ has a neutral cut free proof. So does $\Gamma \vdash \tau C$ and thus $\Gamma \in [C]^\tau_\rho$. Hence Γ is an element of all $[C]^\tau_\rho$ and therefore in their intersection $\check{\bot}$.
- Let $\Gamma \in [A]^\sigma_\phi\ \check{\wedge}\ [B]^\sigma_\phi = [A]^\sigma_\phi \cap [B]^\sigma_\phi$. We have $\Gamma \in [A]^\sigma_\phi$ and $\Gamma \in [B]^\sigma_\phi$ and thus $\Gamma \vdash \sigma A \in [\![A]\!]_\phi$ and $\Gamma \vdash \sigma B \in [\![B]\!]_\phi$. From Definition 12, we get $\Gamma \vdash \sigma(A \wedge B) \in [\![A \wedge B]\!]_\phi$. Hence $\Gamma \in [A \wedge B]^\sigma_\phi$. Conversely, let $\Gamma \in [A \vee B]^\sigma_\phi$, we have $\Gamma \vdash \sigma(A \wedge B) \in [\![A]\!]_\phi \tilde{\wedge} [\![B]\!]_\phi$. If $\Gamma \vdash \sigma(A \wedge B)$ has a neutral and cut free proof, then so do $\Gamma \vdash \sigma A$ and $\Gamma \vdash \sigma B$. Thus $\Gamma \in [A]^\sigma_\phi$ and $\Gamma \in [B]^\sigma_\phi$, hence $\Gamma \in [A]^\sigma_\phi \cap [B]^\sigma_\phi = \Gamma \in [A]^\sigma_\phi \check{\wedge} [B]^\sigma_\phi$. Otherwise we directly have $\Gamma \in [A]^\sigma_\phi$ and $\Gamma \in [B]^\sigma_\phi$ and we conclude the same way.
- Let us first prove $[A]^\sigma_\phi\ \check{\vee}\ [B]^\sigma_\phi \subseteq [A \vee B]^\sigma_\phi$. It is sufficient to prove that $[A \vee B]^\sigma_\phi$ is an upper bound of $[A]^\sigma_\phi$ and $[B]^\sigma_\phi$. Let $\Gamma \in [A]^\sigma_\phi$. We have $(\Gamma \vdash \sigma A) \in [\![A]\!]_\phi$. By Definition 12, $(\Gamma \vdash \sigma(A \vee B)) \in ([\![A]\!]_\phi\ \tilde{\vee}\ [\![B]\!]_\phi) = [\![A \vee B]\!]_\phi$. Thus $\Gamma \in [A \vee B]^\sigma_\phi$. We prove, in a similar way, that if $\Gamma \in [B]^\sigma_\phi$ then $\Gamma \in [A \vee B]^\sigma_\phi$. Conversely, let $\Gamma \in [A \vee B]^\sigma_\phi$. Let C, ρ andτ such that $[A]^\sigma_\phi \cup [B]^\sigma_\phi \subseteq [C]^\tau_\rho$. We have $(\Gamma \vdash \sigma A \vee B) \in ([\![A]\!]_\phi\ \tilde{\vee}\ [\![B]\!]_\phi)$. From Definition 12, there are three cases to consider. First, if $\Gamma \vdash \sigma(A \vee B)$ has a neutral cut free proof. As

$(\Gamma, \sigma A) \in [A]^\sigma_\phi \subseteq [C]^\tau_\rho$, $\Gamma, \sigma A \vdash \tau C$ has a cut free proof by Proposition 4. In a similar way, $\Gamma, \sigma B \vdash \tau C$ has a cut free proof. Hence, we can apply the $\vee$-elim rule and obtain a neutral cut free proof of $\Gamma \vdash \tau C$. Thus $\Gamma \in [C]^\tau_\rho$. As Γ is an element of all such sets $[C]^\tau_\rho$, it is an element of their intersection $[A]^\sigma_\phi \check{\vee} [B]^\sigma_\phi$. Second, if $(\Gamma \vdash \sigma A) \in [\![A]\!]_\phi$. We have $\Gamma \in [A]^\sigma_\phi \subseteq [C]^\tau_\rho$. Again, Γ is an element of the intersection of all such sets. The case $\Gamma \vdash \sigma B \in [\![B]\!]_\phi$ is similar.

– Let us prove $[A \Rightarrow B]^\sigma_\phi \subseteq [A]^\sigma_\phi \check{\Rightarrow} [B]^\sigma_\phi$. This is equivalent to $[A]^\sigma_\phi \cap [A \Rightarrow B]^\sigma_\phi \subseteq [B]^\sigma_\phi$. Suppose $\Gamma \vdash \sigma A \in [\![A]\!]_\phi$ and $\Gamma \vdash \sigma A \Rightarrow \sigma B \in [\![A \Rightarrow B]\!]_\phi = [\![A]\!]_\phi \tilde{\Rightarrow} [\![B]\!]_\phi$. If $\Gamma \vdash \sigma A \Rightarrow \sigma B$ has a neutral cut free proof, so does $\Gamma \vdash \sigma B$, as $\Gamma \vdash \sigma A$ has a cut free proof. Thus $\Gamma \in [B]^\sigma_\phi$. Otherwise, considering an empty Σ, we have $\Gamma \vdash \sigma A \in [\![A]\!]_\phi$ and thus we get $\Gamma \vdash \sigma B \in [\![B]\!]_\phi$. Conversely let us prove $[A]^\sigma_\phi \check{\Rightarrow} [B]^\sigma_\phi \subseteq [A \Rightarrow B]^\sigma_\phi$. We have to prove that $[A \Rightarrow B]^\sigma_\phi$ is an upper bound of the set of all the c such that $c \cap [A]^\sigma_\phi \subseteq [B]^\sigma_\phi$. Let such a c, we have to prove $c \subseteq [A \Rightarrow B]^\sigma_\phi$. As noticed c has the form $\bigcap [C_i]^{\tau_i}_{\rho_i}$. Let $\Gamma \in c$. We must show $\Gamma \vdash \sigma A \Rightarrow \sigma B \in [\![A \Rightarrow B]\!]_\phi = [\![A]\!]_\phi \tilde{\Rightarrow} [\![B]\!]_\phi$. For this, let Σ such that $\Gamma, \Sigma \vdash \sigma A \in [\![A]\!]_\phi$. This is equivalent to $\Gamma, \Sigma \in [A]^\sigma_\phi$. By Proposition 4, we know that $\Gamma, \Sigma \in [C_i]^{\tau_i}_{\rho_i}$. Therefore it is an element of their intersection. Thus $\Gamma, \Sigma \in [B]^\sigma_\phi$. Finally $(\Gamma, \Sigma \vdash \sigma B) \in [\![B]\!]_\phi$. Hence $c \subseteq [A \Rightarrow B]^\sigma_\phi$.

– Let $\Gamma \in \bigcap \{[A]^{\sigma+(t/x)}_{\phi+(d/x)}\}$. Then we have for any t and any d, $\Gamma \vdash (\sigma+(t/x))A \in [\![A]\!]_{\phi+(d/x)}$. Hence, $\Gamma \vdash \sigma \forall x A \in \tilde{\forall}\{[\![A]\!]_{\phi+(d/x)}\} = [\![\forall x A]\!]_\phi$. Conversely, let $\Gamma \in [\forall x A]^\sigma_\phi$. Then $\Gamma \vdash \sigma \forall x A \in [\![\forall x A]\!]_\phi$. If $\Gamma \vdash \sigma \forall x A$ has a neutral cut free proof then so does the sequent $\Gamma \vdash (\sigma + (t/x))A$ for any t and this sequent is an element of $[\![A]\!]_{\phi+(d/x)}$ for any d. Hence $\Gamma \in [A]^{\sigma+(t/x)}_{\phi+(d/x)}$ for any t, d. So it is an element of their intersection. Otherwise, by Definition 12, for all t and d we have $\Gamma \vdash (\sigma + (t/x))A \in [\![A]\!]_{\phi+(d/x)}$ thus it is an element of their intersection.

– We first prove that for any t, d, $[\exists x A]^\sigma_\phi$ is an upper bound of $\{[A]^{\sigma+(t/x)}_{\phi+(d/x)} \mid t \in \mathcal{T}, d \in \mathcal{M}\}$. Let t, d and $\Gamma \in [A]^{\sigma+(t/x)}_{\phi+(d/x)}$. We have $\Gamma \vdash (\sigma + (t/x))A \in [\![A]\!]_{\phi+(d/x)}$. By Definition 12, $\Gamma \vdash \sigma \exists x A \in \tilde{\exists}\{[\![A]\!]_\phi\}$. Hence $\Gamma \in [\exists x A]^\sigma_\phi$ and $[A]^{\sigma+(t/x)}_{\phi+(d/x)} \subseteq [\exists x A]^\sigma_\phi$ for any t, d so $\check{\exists}\ \{[A]^{\sigma+(t/x)}_{\phi+(d/x)} \mid t \in \mathcal{T}, d \in \mathcal{M}\} \subseteq [\exists x A]^\sigma_\phi$. Conversely, let $\Gamma \in [\exists x A]^\sigma_\phi$. Suppose $\Gamma \vdash \sigma \exists x A$ has a neutral cut free proof. Let $u = \bigcap [C_i]^{\tau_i}_{\rho_i}$ an upper bound of $\{[A]^{\sigma+(t/x)}_{\phi+(d/x)} \mid t \in \mathcal{T}, d \in \mathcal{M}\}$. We can choose $u = [C]^\tau_\rho$, since we need the intersection of the upper bounds. Since this holds for any $t \in \mathcal{T}$, $d \in \mathcal{M}$ this also holds for a variable y not appearing in ϕ and σ. Let $\phi' = \phi + (d/x)$ and $\sigma' = \sigma + (y/x)$. By definition: $[A]^{\sigma'}_{\phi'} \subseteq [C]^\tau_\rho$. By Proposition 4, $\sigma' A \in [A]_{\phi'}$. Hence $\sigma' A \in [C]^\tau_\rho$. Thus the sequent $\sigma' A \vdash \tau C \in [\![C]\!]_\rho$ has a cut free proof by Proposition 4. Thus so does the sequent $\Gamma \vdash \tau C$. Hence $\Gamma \in [C]^\tau_\rho$. This is valid for any C upper bound of $\{[A]^{\sigma+(t/x)}_{\phi+(d/x)} \mid t \in \mathcal{T}, d \in \mathcal{M}\}$. So, Γ in their least upper bound $\tilde{\exists}\{[A]^{\sigma+(t/x)}_{\phi+(d/x)}\}$ Otherwise by Definition 12, $\Gamma \vdash \sigma \exists x A$ is such that for some term t and element d, $\Gamma \vdash (\sigma + (t/x))A \in [\![A]\!]_{\phi+(d/x)}$. This shows that $\Gamma \in [A]^{\sigma+(t/x)}_{\phi+(d/x)}$. Then $\Gamma \in \tilde{\exists}\{[A]^{\sigma+(t/x)}_{\phi+(d/x)} \mid t \in \mathcal{T}, d \in \mathcal{M}\}$.

3.3 Cut elimination

Let $\Gamma = A_1, ..., A_n$ be a context, we write $[\Gamma]^\sigma_\phi$ for $[A_1]^\sigma_\phi \check{\wedge} ... \check{\wedge} [A_n]^\sigma_\phi$.

Proposition 7. $[(t/x)A]^\sigma_\phi = [A]^{\sigma+(\sigma t/x)}_{\phi+([\![t]\!]_\phi/x)}$.

Proof. $\Gamma \in [(t/x)A]^\sigma_\phi$ if and only if $(\Gamma \vdash \sigma((t/x)A)) \in [\![(t/x)A]\!]_\phi$ if and only if $(\Gamma \vdash (\sigma + (\sigma t/x))A) \in [\![A]\!]_{\phi+([\![t]\!]_\psi/x)}$ if and only if $\Gamma \in [A]^{\sigma+(\sigma t/x)}_{\phi+([\![t]\!]_\phi/x)}$.

Proposition 8. *If $\Gamma \vdash B$ is provable then for every substitution σ, every valuation ϕ, $[\Gamma]^\sigma_\phi \subseteq [B]^\sigma_\phi$.*

Proof. By induction on the structure of the proof of $\Gamma \vdash B$.

- If the last rule is the axiom rule, then the formula B is equivalent to one of the A_i's and thus $[\Gamma]^\sigma_\phi \subseteq [B]^\sigma_\phi$.
- If the last rule is $\top$-intro then $B \equiv \top$ and as Ω is a Heyting algebra $[\Gamma]^\sigma_\phi \subseteq \check{\top} = [B]^\sigma_\phi$.
- If the last rule is $\bot$-elim, then the premise is equivalent to $\Gamma \vdash \bot$. By induction hypothesis, we have $[\Gamma]^\sigma_\phi \subseteq [\bot]^\sigma_\phi = \check{\bot}$. Hence, as Ω is a Heyting algebra, $[\Gamma]^\sigma_\phi \subseteq [B]^\sigma_\phi$.
- If the last rule is $\wedge$-intro, then $B \equiv (C \wedge D)$ and the premises are equivalent to $\Gamma \vdash C$ and $\Gamma \vdash D$. By induction hypothesis, we have $[\Gamma]^\sigma_\phi \subseteq [C]^\sigma_\phi$ and $[\Gamma]^\sigma_\phi \subseteq [D]^\sigma_\phi$. Thus, as Ω is a Heyting algebra, $[\Gamma]^\sigma_\phi \subseteq [D]^\sigma_\phi \check{\wedge} [D]^\sigma_\phi = [C \wedge D]^\sigma_\phi = [B]^\sigma_\phi$.
- If the last rule is $\wedge$-elim1 then the premise is equivalent to $\Gamma \vdash B \wedge C$ and by induction hypothesis, we have $[\Gamma]^\sigma_\phi \subseteq [B \wedge C]^\sigma_\phi = ([B]^\sigma_\phi \check{\wedge} [C]^\sigma_\phi)$. Thus, as Ω is a Heyting algebra, $[\Gamma]^\sigma_\phi \subseteq [B]^\sigma_\phi$.
- If the last rule is $\wedge$-elim2, the proof is similar.
- If the last rule is $\vee$-intro1, then $B \equiv (C \vee D)$ and the premise is equivalent to $\Gamma \vdash C$. By induction hypothesis, we have $[\Gamma]^\sigma_\phi \subseteq [C]^\sigma_\phi$. Thus, as Ω is a Heyting algebra, $[\Gamma]^\sigma_\phi \subseteq [C]^\sigma_\phi \check{\vee} [D]^\sigma_\phi = [C \vee D]^\sigma_\phi = [B]^\sigma_\phi$.
- If the last rule is $\vee$-intro2, the proof is similar.
- If the last rule is $\vee$-elim then the premises are equivalent to $\Gamma \vdash C \vee D$, $\Gamma, C \vdash B$ and $\Gamma, D \vdash B$. By induction hypothesis, we have $[\Gamma]^\sigma_\phi \subseteq ([C \vee D]^\sigma_\phi) = ([C]^\sigma_\phi \check{\vee} [D]^\sigma_\phi)$, $[\Gamma]^\sigma_\phi \check{\wedge} [C]^\sigma_\phi \subseteq [B]^\sigma_\phi$ and $[\Gamma]^\sigma_\phi \check{\wedge} [D]^\sigma_\phi \subseteq [B]^\sigma_\phi$. Thus, as Ω is a Heyting algebra, $[\Gamma]^\sigma_\phi \subseteq [B]^\sigma_\phi$.
- If the last rule is $\Rightarrow$-intro, then $B \equiv (C \Rightarrow D)$ and the premise is equivalent to $\Gamma, C \vdash D$. By induction hypothesis, we have $[\Gamma]^\sigma_\phi \check{\wedge} [C]^\sigma_\phi \subseteq [D]^\sigma_\phi$. Thus, as Ω is a Heyting algebra, $[\Gamma]^\sigma_\phi \subseteq [C]^\sigma_\phi \check{\Rightarrow} [D]^\sigma_\phi = [C \Rightarrow D]^\sigma_\phi = [B]^\sigma_\phi$.
- If the last rule is $\Rightarrow$-elim, then the premises are equivalent to $\Gamma \vdash C \Rightarrow B$ and $\Gamma \vdash C$. By induction hypothesis, we have $[\Gamma]^\sigma_\phi \subseteq [C \Rightarrow B]^\sigma_\phi = ([C]^\sigma_\phi \check{\Rightarrow} [B]^\sigma_\phi)$ and $[\Gamma]^\sigma_\phi \subseteq [C]^\sigma_\phi$. Thus, as Ω is a Heyting algebra, $[\Gamma]^\sigma_\phi \subseteq [B]^\sigma_\phi$.

- If the last rule is $\forall$-intro, then $B \equiv \forall x\ C$, the premise is equivalent to $\Gamma \vdash C$ and x does not occur in Γ. By induction hypothesis, we have for all t and d, $[\Gamma]^{\sigma}_{\phi} \subseteq [C]^{\sigma+(x,t)}_{\phi+(x,d)}$. Thus, as Ω is a Heyting algebra, $[\Gamma]^{\sigma}_{\phi} \subseteq \check{\forall}\ \{[C]^{\sigma+(t/x)}_{\phi+(d/x)} \mid t \in \mathcal{T}, d \in \mathcal{M}\} = [\forall x\ C]^{\sigma}_{\phi} = [B]^{\sigma}_{\phi}$.
- If the last rule is $\forall$-elim, then $B \equiv (t/x)C$ and the premise is equivalent to $\Gamma \vdash \forall x\ C$. By induction hypothesis, we have $[\Gamma]^{\sigma}_{\phi} \subseteq [\forall x\ C]^{\sigma}_{\phi} = \check{\forall}\ \{[C]^{\sigma+(t/x)}_{\phi+(d/x)} \mid t \in \mathcal{T}, d \in \mathcal{M}\}$. Thus, as Ω is a Heyting algebra, $[\Gamma]^{\sigma}_{\phi} \subseteq [C]^{\sigma+(\sigma t/x)}_{\phi+(\llbracket t \rrbracket/x)} = [(t/x)C]_{\phi} = [B]^{\sigma}_{\phi}$.
- If the last rule is $\exists$-intro, then $B \equiv \exists x\ C$ and the premise is equivalent to $\Gamma \vdash (t/x)C$. By induction hypothesis, we have $[\Gamma]^{\sigma}_{\phi} \subseteq [(t/x)C]^{\sigma}_{\phi} = [C]^{\sigma+(\sigma t/x)}_{\phi+(\llbracket t \rrbracket/x)}$. Thus, as Ω is a Heyting algebra, $[\Gamma]^{\sigma}_{\phi} \subseteq \check{\exists}\ \{[C]^{\sigma+(t/x)}_{\phi+(d/x)} \mid t \in \mathcal{T}, d \in \mathcal{M}\} = [\exists x\ C]^{\sigma}_{\phi} = [B]^{\sigma}_{\phi}$.
- If the last rule is $\exists$-elim then the premises are equivalent to $\Gamma \vdash \exists x\ C$ and $\Gamma, C \vdash B$ where the variable x occurs neither in Γ nor in B. By induction hypothesis, we have $[\Gamma]^{\sigma}_{\phi} \subseteq [\exists x\ C]^{\sigma}_{\phi} = \check{\exists}\ \{[C]^{\sigma+(t/x)}_{\phi+(d/x)} \mid t \in \mathcal{T}, d \in \mathcal{M}\}$ and for all t and d, $[\Gamma]^{\sigma}_{\phi}\ \check{\wedge}\ [C]^{\sigma+(t/x)}_{\phi+(d/x)} \subseteq [B]^{\sigma}_{\phi}$. Thus, as Ω is a Heyting algebra, $[\Gamma]^{\sigma}_{\phi} \subseteq [B]^{\sigma}_{\phi}$.

Theorem 1 (Cut elimination). *If the sequent $\Gamma \vdash B$ is provable, then it has a cut free proof.*

Proof. Let $\Gamma = A_1, ..., A_n$. Consider an arbitrary valuation ϕ and an arbitrary substitution σ. By Proposition 4, we have $\Gamma \in [A_1]^{\sigma}_{\phi}$, ..., $\Gamma \in [A_n]^{\sigma}_{\phi}$, thus $\Gamma \in [\Gamma]^{\sigma}_{\phi}$. By Proposition 8, we have $\Gamma \in [B]^{\sigma}_{\phi}$, and, by Proposition 4 again, $\Gamma \vdash B$ has a cut free proof.

4 Hybridization

In this section, we show that the set of contexts $[A]^{\sigma}_{\phi}$ can be seen as the denotation of the proposition A in another model $\mathcal{D}$ that is Ω-valued and build from $\mathcal{M}$. This model will permit to give an alternative cut elimination proof that proceeds by proving the completeness of the cut free calculus. The elements of this model are quite similar to the V-complexes used in the proofs of cut elimination for simple type theory that proceed by proving the completeness of the cut free calculus. So we shall give a generalization of the notion of V-complex that can be used not only for simple type theory but for all super-consistent theories.

Consider a super-consistent theory and the its model $\mathcal{M}$ defined in section 3.

Definition 16 (The model $\mathcal{D}$). *The model $\mathcal{D}$ is a Ω-valued model with domain $\mathcal{D} = \mathcal{T}' \times \mathcal{M}$ where $\mathcal{T}'$ is the set of (classes modulo $\equiv$ of) open terms of the language of the theory and $\mathcal{M}$ the domain of the model $\mathcal{M}$.*

Let f be a function symbol of the language and $\hat{f}^{\mathcal{M}}$ its interpretation in the model $\mathcal{M}$, the interpretation $\hat{f}^{\mathcal{D}}$ of this symbol in the model $\mathcal{D}$ is the function from $\mathcal{D}^n$ to $\mathcal{D}$

$$\langle t_1, a_1\rangle, ..., \langle t_n, a_n\rangle \mapsto \langle f(t_1, ..., t_n), \hat{f}^{\mathcal{M}}(a_1, ..., a_n)\rangle$$

Let P be a predicate symbol of the language and $\hat{P}^{\mathcal{M}}$ its interpretation in the model $\mathcal{M}$. The interpretation $\hat{P}^{\mathcal{D}}$ of this symbol in the model $\mathcal{D}$ is the function from $\mathcal{D}^n$ to Ω

$$\langle t_1, a_1\rangle, ..., \langle t_n, a_n\rangle \mapsto \hat{P}^{\mathcal{M}}(a_1, ..., a_n) \lhd P(t_1, ..., t_n)$$

Let ψ be an assignment mapping variables to elements $\langle t, d\rangle$ of $\mathcal{D}$. We write ψ^1 for the substitution mapping the variable x to a fixed representative of the first component of ψx and ψ^2 for the $\mathcal{M}$-assignment mapping x to the second component of ψx. Notice that by proposition 4, $[A]_{\psi^2}^{\psi^1}$ is independent of the choice of the representatives in ψ^1.

Proposition 9. *For any term t and assignment ψ*

$$[\![t]\!]_{\psi}^{\mathcal{D}} = \langle \psi^1 t, [\![t]\!]_{\psi^2}^{\mathcal{M}}\rangle$$

For any proposition A and assignment ψ

$$[\![A]\!]_{\psi}^{\mathcal{D}} = [A]_{\psi^2}^{\psi^1}$$

Proof. The first statement is proved by induction on the structure of t. The second by induction over the structure of A. If A is atomic, the result follows from the first statement, Definition 13 and Definition 16 and in all the other cases, from Proposition 6 and the induction hypothesis.

Proposition 10 ($\mathcal{D}$ is a model of $\equiv$). *If $A \equiv B$, then $[\![A]\!]_{\psi}^{\mathcal{D}} = [\![B]\!]_{\psi}^{\mathcal{D}}$.*

Proof. From Proposition 4, we have $[A]_{\psi^2}^{\psi^1} = [B]_{\psi^2}^{\psi^1}$, and, by Proposition 9, we get $[\![A]\!]_{\psi}^{\mathcal{D}} = [\![B]\!]_{\psi}^{\mathcal{D}}$.

We obtain this way an alternative proof of Proposition 8 and hence of the cut elimination theorem. Indeed, using Proposition 9, this rephrases to: if $\Gamma \vdash B$ then $[\![\Gamma]\!]_{\psi}^{\mathcal{D}} \subseteq [\![B]\!]_{\psi}^{\mathcal{D}}$ and this is just the soundness theorem. The proof of Proposition 8 is indeed proof of an instance of the soundness theorem.

Unlike the model $\mathcal{M}$ where all propositions are valid (because all truth values of $\mathcal{S}$ are positive), the model $\mathcal{D}$ allows to formulate the cut elimination proof as a consequence of the completeness of the cut free calculus.

Proposition 11 (Completeness of the cut free calculus). *Let $A_1, ..., A_n, B$ be propositions. If the proposition $(A_1 \wedge ... \wedge A_n) \Rightarrow B$ is valid in all Heyting algebra valued models of the theory, then the sequent $A_1, ..., A_n \vdash B$ has a cut free proof.*

Proof. The model $\mathcal{D}$ is a Heyting algebra valued model of the theory, thus the proposition $(A_1 \wedge ... \wedge A_n) \Rightarrow B$ is valid in $\mathcal{D}$, *i.e.* for all ψ, $\bigcap [\![A_i]\!]^{\mathcal{D}}_{\psi} \subseteq [\![B]\!]^{\mathcal{D}}_{\psi}$, *i.e.*, by Proposition 9, $\bigcap [A_i]^{\psi^1}_{\psi^2} \subseteq [B]_{\psi}$. But, by Proposition 4, $A_1, ..., A_n \in [A_i]^{\psi^1}_{\psi^2}$. Hence $A_1, ..., A_n \in [B]^{\psi^1}_{\psi^2}$. Therefore, by Proposition 4, the sequent $A_1, ..., A_n \vdash B$ has a cut free proof.

5 Application to simple type theory

As a particular case, we get a cut elimination proof for simple type theory.

Let us detail the models constructions in this case. Based on the language of simple type theory, we first build the truth values algebra of sequents $\mathcal{S}$ as in Definition 12. Then using the super-consistency of simple type theory, we build the model $\mathcal{M}$ as in Proposition 2. In particular, we let $\mathcal{M}_\iota = \{0\}$, $\mathcal{M}_o = \mathcal{S}$, and $\mathcal{M}_{T \to U} = \mathcal{M}_U^{\mathcal{M}_T}$. Then we let $\mathcal{D}_T = \mathcal{T}_T \times \mathcal{M}_T$, where $\mathcal{T}_T$ is the set of terms of sort T. In particular, we have $\mathcal{D}_\iota = \mathcal{T}_\iota \times \{0\}$ and $\mathcal{D}_o = \mathcal{T}_o \times \mathcal{S}$. See [3] for further details.

This construction is reminiscent of the definition of V-complexes, that are also ordered pairs whose first component is a term. In particular, in the definition of V-complexes of [10, 11, 1, 2, 7], we also take $\mathcal{C}_\iota = \mathcal{T}_\iota \times \{0\}$ but $\mathcal{C}_o = \mathcal{T}_o \times \{\mathbf{false}, \mathbf{true}\}$. In the intuitionistic case [2], we replace $\{\mathbf{false}, \mathbf{true}\}$ by a complete Heyting algebra. The semantic meaning is the second component.

The difference here is that instead of using the small algebra $\{\mathbf{false}, \mathbf{true}\}$ or a Heyting algebra, we use the larger truth values algebra $\mathcal{S}$ of sequents.

Another difference is that, in our definition, we first define completely the hierarchy $\mathcal{M}$ and then perform the hybridization with the terms. Terms and functions are more intricate in the definition of V-complexes as $\mathcal{C}_{T \to U}$ is defined as a set of pairs formed with a term t of type $T \to U$ and a function f from $\mathcal{C}_T$ to $\mathcal{C}_U$ such that $f\langle t', f' \rangle$ is a pair whose first component is tt'. In our definition, in contrast, $\mathcal{D}_{T \to U}$ is the set of pairs formed with a term t of type $T \to U$ and and an element of $\mathcal{M}_{T \to U}$, *i.e.* a function from $\mathcal{M}_T$ to $\mathcal{M}_U$, not from $\mathcal{D}_T$ to $\mathcal{D}_U$. When we apply a pair $\langle t, f \rangle$ of $\mathcal{D}_{T \to U}$ to a pair $\langle t', f' \rangle$ of $\mathcal{D}_T$, we just apply componentwise t to t' and f to f' and get the pair $\langle tt', ff' \rangle$. With the usual V-complexes, in contrast, the result of application is the pair $f\langle t', f' \rangle$ whose first component is indeed tt', but whose second component depends on f, f', and also t'. This is indeed necessary since, in the algebra $\{\mathbf{false}, \mathbf{true}\}$ or in a Heyting algebra, $\top$ and $\top \wedge \top$ have the same interpretation and thus in the usual V-complexes models, the interpretation of $\top$ and $\top \wedge \top$ have the same second component. The only way to make the second component of $P(\top)$ and $P(\top \wedge \top)$ different is to make it depending of the first component. In our truth value algebra, in contrast, $\top$ and $\top \wedge \top$ have different interpretations. Moreover, $[\![P(\top)]\!]^{\mathcal{D}} = [\![P(\top)]\!]^{\mathcal{M}} \lhd P(\top)$ whereas $[\![P(\top \wedge \top)]\!]^{\mathcal{D}} = [\![P(\top \wedge \top)]\!]^{\mathcal{M}} \lhd P(\top \wedge \top)$. In the first case, the element contains $P(\top)$ since $P(\top) \vdash P(\top)$ has a neutral cut free proof and $[\![P(\top)]\!]^{\mathcal{M}}$ is a member of the algebra of sequents. In the second case, it contains $P(\top \wedge \top)$ for the same reason. But the converse is not true. Hence these truth values are distinct.

Thus the main difference between our hybrid model construction and that of the V-complexes is that we have broken this dependency of the right component of the pair obtained by applying $\langle t, f\rangle$ to $\langle t', f'\rangle$ with respect to t' leading to a simpler construction in two steps. The reason why we have been able to do so, is that starting with a larger algebra for $\mathcal{D}_o$, our semantic components are more informative and thus is sufficient to define the interpretation of larger terms. Once this problem, specific to simple type theory, is solved, the V-complex construction boils down to hybridization and can be generalized, as we have seen, to all super-consistent theories.

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